The probatilistic Quantifier Fuzzification Mechanism $\mathcal{F}^A$: A theoretical analysis.

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Abstract

The main goal of this work is to analyze the behaviour of the $\mathcal{F}^A$ quantifier fuzzification mechanism [23, 22, 17]. As we prove in the paper, this model has a very solid theorethical behaviour, superior to most of the models defined in the literature. Moreover, we show that the underlying probabilistic interpretation has very interesting consequences.

Keywords: Quantifier fuzzification mechanism, Determiner fuzzification schemes, Probabilistic quantification models

1 Introduction

The evaluation of fuzzy quantified expressions is a topic that has been widely dealt with in literature [2, 7, 8, 13, 14, 15, 16, 20, 24, 25, 28, 30, 36, 32, 31, 33, 34, 40, 56, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 58] since the use of quantified expressions in fields such as fuzzy control [50], temporal reasoning in robotics, [11, 10, 44, 43], complex fuzzy queries in databases [8, 9], information retrieval [6, 5, 41, 23, 22, 35], data fusion [51, 37], etc. can take advantage of using vague and interpretable quantification models. Moreover, the definition of adequate models to evaluate quantified expressions is fundamental to perform "computing with words", topic that was suggested by Zadeh [59] to express the ability of programming systems in a linguistic way. In this paper we analyze the theoretical behavior and some practical consequences of the $\mathcal{F}^A$ model defined on [23, 22][1]. Furthermore, we show that the underlying probabilistic interpretation of this model hints the utility of the model for a number of applications.

In general, approaches to fuzzy quantification in the literature use the concept of *fuzzy linguistic quantifier* [58] to represent absolute or proportional fuzzy quantities. Zadeh [58] defines *quantifiers of the first kind* as quantifiers used for representing absolute quantities (defined by using fuzzy numbers on $\mathbb{N}$) , and *quantifiers of the second kind* as quantifiers used for representing relative quantities (defined by using fuzzy numbers on $[0, 1]$). In the literature, quantifiers of

[1] Most of the theoretical results presented in this paper have been previously published in the dissertation [17], in spanish.

the first kind are associated to sentences involving only one single fuzzy property (as in *"about three men are tall"* where *"tall"* is a fuzzy property); and quantifiers of the second kind are associated to sentences involving two fuzzy properties (as in *"about 70% of blond men are tall"* where *"blond"* and *"tall"* are fuzzy properties). The linguistic quantifier associated to the former sentence denotes the semantics of *"about 3"* and is defined by using a fuzzy number with domain on $\mathbb{N}$. The linguistic quantifier associated to the second sentence represents the semantics of *"about 70%"* and is defined by using a fuzzy number with domain on $[0, 1]$.

Moreover, most of the existing approaches for dealing with fuzzy quantification are based on the evaluation of the compatibility between the linguistic quantifier and a scalar, possibilistic or probabilistic cardinality measure for the involved fuzzy sets. Scalar approaches [58], usually consist of a simple evaluation of the quantifier on the cardinality value. For possibilistic approaches, an overlapping measure *SUP-min* is generally used [14, 16, 45] whilst for probabilistic approaches, [15, 16, 21] a weighted mean of all the compatibility values is computed. OWA approaches [51, 54] can also be related to the probabilistic interpretation. A different approach is used in [28, 30, 36, 32, 31, 33, 34], where families of models that are based on a three valued interpretation of fuzzy sets are defined.

For analyzing the behavior of fuzzy quantification models different properties of convenient or necessary fulfillment have been defined [16, 28, 30, 36, 32, 31, 33, 34, 46]. Most of the approaches in literature fail to exhibit a plausible behavior [2, 16, 29, 32, 33, 17, 46], and only a few [16, 21, 28, 30, 36, 32, 31, 33] seem to exhibit an adequate behavior in the general case.

In this work we will follow the Glöckner approximation to fuzzy quantification [28, 30, 36, 32, 31, 33, 34]. In his approach, the author generalizes the concept of *generalized classic quantifier* [3, 27, 38] (second order predicates or set relationships) to the fuzzy case; that is, a *fuzzy quantifier* is a fuzzy relationship between fuzzy sets. And then rewrites the fuzzy quantification problem as the problem of looking for mechanism to transform *semi-fuzzy quantifiers* (quantifiers between generalized classic quantifiers and fuzzy quantifiers that are adequate to specify the meaning of quantified expressions) to fuzzy quantifiers.

Moreover, Glöckner has also defined a rigorous axiomatic framework to assure the good behavior of QFMs. Models fulfilling this framework are called *Determiner fuzzification schemes (DFSs)* and they fulfill an important set of appropriate behavior properties.

The main goal of this work is to analyze the behavior of the $\mathcal{F}^A$ model [23, 22, 17]. This model has a very solid theoretical behavior, superior to most of the models defined in the literature. Moreover, we show that the underlying probabilistic interpretation based on likelihood functions [42, 47, 4, 26] has very interesting consequences, that assure its utility for a number of applications. For example, in [23, 22, 17] the application of the model in a information retrieval task was shown, with competitive results. In [18] the model has been used in a summarization application for the evaluation of quantified temporal expressions. From a theoretical point of view the model is a DFS, although is only defined in finite domains. The fulfillment of the DFS axioms guarantees a very good theoretical behavior. As an important point, the fuzzy operators induced by the model are the product t-norm and the probabilistic sum t-conorm. This fact makes the $\mathcal{F}^A$ model essentially different of the models defined in [33] because all those "standard models" induce the min tnorm and the max tconorm. To our knowledge, the $\mathcal{F}^A$ model is the unique known non standard DFSs.

The paper is organized as follows. In the first section, we resume the Glöckner's approach to fuzzy quantification, based on quantifier fuzzification mechanisms[2]. In the second section we explain some of the properties that let us to analyze the behavior of the quantification model. Most of them are a compilation of the properties defined on [33, 34, chapters 3 and 4], but we have added to these properties two very interesting properties fulfilled by the $\mathcal{F}^A$ model and by the probabilistic models defined in [21]. In section three the $\mathcal{F}^A$ QFM is defined. We also explores the behavior of the model when the cardinality of the base set tends to infinite, with a surprising relation with original $\sum Count$ Zadeh's model [58]. Next section is devoted to some interesting consequences of the probabilistic interpretation of the $\mathcal{F}^A$ QFM, with relation with a number of application fields. Proofs of the properties and efficient algorithm solutions are collected in two apendixes. A bibliographic analysis of quantification models has not been included as it can be found in [2, 16, 29, 32, 33, 17, 46].

2 Quantifier fuzzification mechanisms

To overcome the Zadeh's framework to fuzzy quantification Glöckner [34] rewrites the problem of fuzzy quantification as the problem of looking for adequate means to convert the specification means (semi-fuzzy quantifiers) into the operational means (fuzzy quantifiers) [34]. In this section we explain in some detail the framework proposed by Glöckner to achieve that result.

Fuzzy quantifiers are just a fuzzy generalization of crisp or classic quantifiers. Before giving the definition of fuzzy quantifiers, we will show the definition of classic quantifiers and some examples:

Definition 1 (Classic quantifier.) *[34, pag. 57] A two valued (generalized) quantifier on a base set* $E \neq \varnothing$ *is a mapping* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$*, where* $n \in \mathbb{N}$ *is the arity (number of arguments) of* Q*,* $\mathbf{2} = \{0, 1\}$ *denotes the set of crisp truth values, and* $\mathcal{P}(E)$ *is the powerset of* E*.*

In this work we assume the base set E is finite as the $\mathcal{F}^A$ model is only defined on finite base sets.

[2] A complete explanation of the QFM framework can be consulted in the excellent work [34].

Examples of some definitions of classic quantifiers are:

$$\mathbf{all}\left(Y_1, Y_2\right) = Y_1 \subseteq Y_2 \tag{1}$$

$$\mathbf{at\_least}80\%\left(Y_1, Y_2\right) = \begin{cases} \frac{|Y_1 \cap Y_2|}{|Y_1|} \geq 0.80 & X_1 \neq \varnothing \\ 1 & X_1 = \varnothing \end{cases}$$

Example 2 *Let us consider the evaluation of the sentence "at least eighty percent of the members are lawyers" where the properties "members" and "lawyers" are respectively defined as* $Y_1 = \{1,0,1,0,1,0,1,1\}, Y_2 = \{1,0,1,0,1,0,0,0\}$, *and "at least eighty percent" is defined in expression 1. Then* $\mathbf{at\_least}80\%\left(Y_1, Y_2\right) = 0$.

In a fuzzy quantifier arguments and result can be fuzzy. The definition of a fuzzy quantifier is:

Definition 3 (Fuzzy Quantifier) *[34, pag. 66] An n-ary fuzzy quantifier* $\widetilde{Q}$ *on a base set* $E \neq \varnothing$ *is a mapping* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I} = [0,1]$. *Here* $\widetilde{\mathcal{P}}(E)$ *denotes the fuzzy powerset of* E.

A fuzzy quantifier assigns a gradual result to each choice of $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

An example of a fuzzy quantifier could be $\widetilde{\mathbf{all}} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$. A reasonable fuzzy definition of the fuzzy quantifier $\widetilde{\mathbf{all}}$ is:

$$\widetilde{\mathbf{all}}\left(X_1, X_2\right) = \inf\left\{\max\left(1 - \mu_{X_1}(e), \mu_{X_2}(e)\right) : e \in E\right\} \tag{2}$$

Example 4 *Let us consider the evaluation of the sentence "all big houses are overvalued" in a referential set* $E = \{e_1, \ldots, e_4\}$. *Let us assume that properties "big" and "overvalued" are respectively defined as:* $X_1 = \{0.8/e_1, 1/e_2, 0.6/e_3, 0.3/e_4\}$, $X_2 = \{0.9/e_1, 0.7/e_2, 0.3/e_3, 0.2/e_4\}$. *If we use expression (2) then:* $\widetilde{\mathbf{all}}\left(X_1, X_2\right) = \inf\left\{\max\left(1 - \mu_{X_1}(e), \mu_{X_2}(e)\right) : e \in E\right\} = 0.4$.

Although a certain consensus may be achieved to accept this previous expression as a suitable definition for $\widetilde{\mathbf{all}}$ this is not the unique one. The problem of establishing consistent fuzzy definitions for quantifiers (e.g., *"at least eighty percent"*) is faced in [34] by introducing the concept of semi-fuzzy quantifiers. A semi-fuzzy quantifier represents a medium point between classic quantifiers and fuzzy quantifiers, and it is close but is far more general than the idea of Zadeh's linguistic quantifiers [58]. A semi-fuzzy quantifier only accepts crisp arguments, as classic quantifiers, but lets the result range on the truth grade scale $\mathbf{I}$, as for fuzzy quantifiers[3].

Definition 5 (Semi-fuzzy quantifier) *[34, pag. 71] An n-ary semi-fuzzy quantifier* Q *on a base set* $E \neq \varnothing$ *is a mapping* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.

[3] An interesting classification of semi-fuzzy quantifiers is shown in [19]. In [17, chapter 4] an extended classification is defined.

Q assigns a gradual result to each pair of crisp sets $(Y_1,\ldots,Y_n)$.

Examples of semi-fuzzy quantifiers are:

$$\mathbf{about\_5}\left(Y_1,Y_2\right)=T_{2,4,6,8}\left(\left|Y_1\cap Y_2\right|\right) \tag{3}$$

$$\mathbf{at\_least\_about}80\%\left(Y_1,Y_2\right)=\begin{cases} S_{0.5,0.8}\left(\frac{\left|Y_1\cap Y_2\right|}{\left|Y_1\right|}\right) & X_1\neq\varnothing \\ 1 & X_1=\varnothing\end{cases}$$

where $T_{2,4,6,8}\left(x\right)$ and $S_{0.5,0.8}\left(x\right)$ are shown in figure (1)[4].

Example 6 *Let us consider the evaluation of the sentence "about at least 80% the students are Spanish". Let us assume that properties "students" and "Spanish" are respectively defined as:* $Y_1=\{1,0,1,0,1,0,1,1\}, Y_2=\{1,0,1,0,1,0,0,0\}$, *then* $\mathbf{at\_least\_about}80\%\left(Y_1,Y_2\right)=S_{0.5,0.8}\left(\frac{\left|Y_1\cap Y_2\right|}{\left|Y_1\right|}\right)=0.22$.

Semi-fuzzy quantifiers are much more intuitive and easier to define than fuzzy quantifiers, but they do not solve the problem of evaluating fuzzy quantified sentences.

In order to do so mechanisms are needed that enable us to transform semi-fuzzy quantifiers into fuzzy quantifiers, i.e., mappings with domain in the universe of semi-fuzzy quantifiers and range in the universe of fuzzy quantifiers. Glockner names those mechanisms *quantifier fuzzification mechanisms*.

Definition 7 *[34, pag. 74]A quantifier fuzzification mechanism (QFM)* $\mathcal{F}$ *assigns to each semi-fuzzy quantifier* $Q:\mathcal{P}\left(E\right)^n\rightarrow\mathbf{I}$ *a corresponding fuzzy quantifier* $\mathcal{F}\left(Q\right):\widetilde{\mathcal{P}}\left(E\right)^n\rightarrow\mathbf{I}$ *of the same artity* $n\in\mathbb{N}$ *and on the same base set.*

3 Some properties to guarantee the good behavior of QFMs

Before proceeding to explain the QFM $\mathcal{F}^A$ we will introduce some of the properties that let us to guarantee a good behavior of the QFMs. For the sake

[4] Functions $T_{a,b,c,d}$ and $S_{\alpha,\gamma}$ are defined as

$$T_{a,b,c,d}\left(x\right)=\begin{cases} 0 & x\leq a \\ \frac{x-a}{b-a} & a<x\leq b \\ 1 & b<x\leq c \\ 1-\frac{x-c}{d-c} & c<x\leq d \\ 0 & d<x\end{cases}, S_{\alpha,\gamma}\left(x\right)=\begin{cases} 0 & x<\alpha \\ 2\left(\frac{(x-\alpha)}{(\gamma-\alpha)}\right)^2 & \alpha<x\leq\frac{\alpha+\gamma}{2} \\ 1-2\left(\frac{(x-\gamma)}{(\gamma-\alpha)}\right)^2 & \frac{\alpha+\gamma}{2}<x\leq\gamma \\ 1 & \gamma<x\end{cases}$$

In this work, we will use the following relative definitions for the existential and the universal fuzzy number:

$$\exists\left(x\right)=\begin{cases} 0 & x=0 \\ 1 & x>0\end{cases}, \forall\left(x\right)=\begin{cases} 0 & x<1 \\ 1 & x=1\end{cases}$$

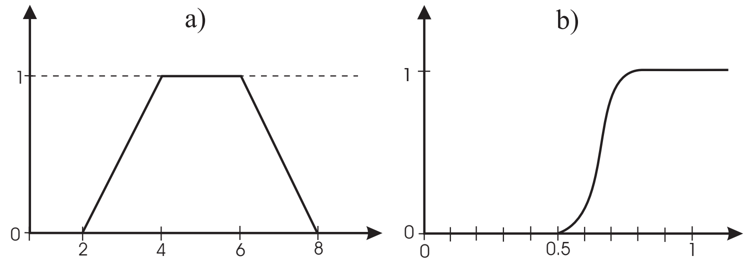

Figure 1: quantifiers **about_5** (a) and **at_least_about_80%** (b)

of brevity, we have only selected some of the more important properties to characterize the behavior of quantification models. A complete and detailed exposition, showing the intuitions under those definitions can be found in [34, chapters three and four.].

The set of properties is organized in three sets. First set is composed of the most important properties that are consequence of the DFS axioms. Second group is composed of some properties that are not consequence of the DFS framework but are important to characterize the behavior of QFMs for different reasons. The last group includes two very important properties that the $\mathcal{F}^A$ model and the probabilistic models defined on [21] fulfills[5].

In the appendix we show the proof of those properties for the $\mathcal{F}^A$ QFM.

3.1 Some properties that are consequence of the DFS axiomatic framework

3.1.1 Correct generalization property (P.1)

Perhaps the most fundamental property to be fulfilled by a QFM is the correct generalization property. This property, defined independently by Glöckner [28] for QFMs and by Delgado et al. for models following the Zadeh's framework [46, 16], requires that the behavior of a fuzzy quantifier $\mathcal{F}(Q)$ on crisp arguments was the expected; that is, the results obtained with a fuzzy quantifier $\mathcal{F}(Q)$ and with the corresponding semi-fuzzy quantifier Q must coincide on crisp arguments.

We show now the definition of the property:

Definition 8 (Property of correct generalization) *[34, pag. 112] Let* $Q : \mathcal{P}(E)^n \to \mathbf{I}, n > 0$ *be an n-ary semi-fuzzy quantifier. We say that a QFM* $\mathcal{F}$ *fulfills the property of correct generalization if for all the crisp subsets* $Y_1, \ldots, Y_n \in \mathcal{P}(E)$*, then it holds* $\mathcal{F}(Q)(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_n)$*.*

For a detailed explanation of this property [34, Sections 3.2. and 4.2.] can be consulted.

[5]One of the models defined in [21] is a generalization of an original proposal of Delgado et al. [15, 16] to semi-fuzzy quantifiers.

For example, given crisp sets $Y_1, Y_2 \in \mathcal{P}(E)$, $Y_1 = student$, $Y_2 = spanish$, then this property guarantees that

$$\mathcal{F}(\mathbf{some})(student, spanish) = \mathbf{some}(student, spanish)$$

In the DFS axiomatic framework it is sufficient to guarantee this property in the unary case.

3.1.2 Membership assessment (P.2)

This property is related with the evaluation of the membership grade of a particular element [34, section 3.3.], and belongs to the set of axioms that are used to characterize the DFSs.

In the classic case, we can define a crisp quantifier $\pi_e : \mathcal{P}(E) \to \mathbf{2}$ that test if the element e belongs to the argument set. In the same way, in the fuzzy case, we can define a fuzzy quantifier $\widetilde{\pi}_e$ that returns the membership grade of e. It is natural to require that a reasonable QFM $\mathcal{F}$ maps π_e to $\widetilde{\pi}_e$.

The formal definitions of π_e and $\widetilde{\pi}_e$ are:

Definition 9 *[34, pag. 88] Let E a base set and $e \in E$. The projection quantifier $\pi_e : \mathcal{P}(E) \to \mathbf{2}$ is defined by $\pi_e(Y) = \chi_Y(e)$ for all $Y \in \mathcal{P}(E)$, where $\chi_Y(e)$ denotes the crisp characteristic funtion of the set Y.*

The corresponding fuzzy definition is:

Definition 10 *[34, pag. 88] Let a base set E be given and $e \in E$. The fuzzy projection quantifier $\widetilde{\pi}_e : \widetilde{\mathcal{P}}(E) \to \mathbf{2}$ is defined by $\widetilde{\pi}_e(Y) = \chi_Y(e)$ for all $Y \in \mathcal{P}(E)$.*

Using these definitions the property that establishes that a QFM $\mathcal{F}$ generalizes the quantifier π_e in the correct way is defined:

Definition 11 (Projection quantifiers) *[34, pag. 89, pag. 112] Let $\mathcal{F}$ a QFM. $\mathcal{F}$ fulfills the property of projection quantifiers if it holds $\mathcal{F}(\pi_e) = \widetilde{\pi}_e$ for $E \neq \varnothing$ and $e \in E$.*

3.1.3 Induced operators (P3)

Glöckner explains that a QFM can be used to transform crisp logical operators into fuzzy operators. For example, logical "or" can be extended by using the following semi-fuzzy quantifier defined on a referential set E composed by two elements ($E = \{e_1, e_2\}$):

$$Q_\vee(X) = \begin{cases} 0 & if\ X = \varnothing \\ 1 & if\ X = \{e_1\} \vee X = \{e_2\} \vee X = \{e_1, e_2\} \end{cases}$$

and in this way is possible to define the fuzzy logical function $\widetilde{\vee}$ that is induced by the fuzzification mechanism $\mathcal{F}$ as

$$\widetilde{\vee}(x_1, x_2) = \widetilde{\mathcal{F}}(\vee)(x_1, x_2) = \mathcal{F}(Q_\vee)(\{x_1/e_1, x_2/e_2\})$$

This construction is shown in [30, 36], [34, Section 3.4]. In [28, Sección 1], [34, Section 4.4] a different construction is shown.

To formally define this property the next bijection $\eta : \mathbf{2}^n \to \mathcal{P}(\{1,\ldots,n\})$ is needed:

$$\eta(x_1,\ldots,x_n) = \{k \in \{1,\ldots,n\} : x_k = 1\}$$

for all $x_1,\ldots,x_n \in \mathbf{2}$. In the fuzzy case the analogous bijection is $\mu_{\widetilde{\eta}(x_1,\ldots,x_n)}(k) = x_k$ for all $x_1,\ldots,x_n \in \mathbf{I}$ and $k \in \{1,\ldots,n\}$.

These bijections are used to transform the fuzzy truth functions (i.e. mappings $\mathbf{2}^n \to \mathbf{I}$) in semi-fuzzy quantifiers $Q_f : \mathcal{P}(\{1,\ldots,n\}) \to \mathbf{I}.$ In the same way fuzzy quantifieres $\widetilde{Q} : \widetilde{\mathcal{P}}(\{1,\ldots,n\}) \to \mathbf{I}$ can be transformed in fuzzy truth functions $\widetilde{f} : \mathbf{I}^n \to \mathbf{I}$.

The definition that let us to transform semi-fuzzy truth function in fuzzy truth functions by means of a QFM is the following:

Definition 12 *[34, pag. 90] Suppose $\mathcal{F}$ is a QFM and $f : \mathbf{2}^n \to \mathbf{I}$ is a mapping (i.e. a 'semi-fuzzy truth funtion') for some $n \in \mathbb{N}$. The semi-fuzzy quantifier $Q_f : \mathcal{P}(\{1,\ldots,n\}) \to \mathbf{I}$ is defined by $Q_f(Y) = f\left(\eta^{-1}(Y)\right)$ for all $Y \in \mathcal{P}(\{1,\ldots,n\})$. In terms of Q_f, the induced fuzzy truth function $\widetilde{\mathcal{F}}(f) : \mathbf{I}^n \to \mathbf{I}$ is defined by*

$$\widetilde{\mathcal{F}}(f)(x_1,\ldots,x_n) = \widetilde{\mathcal{F}}(Q_f)\left(\eta^{-1}(x_1,\ldots,x_n)\right)$$

for all $x_1,\ldots,x_n \in \mathbf{I}$.

The construction allows us to transform the usual crisp logical operators ($\neg$, $\wedge$, $\vee$, $\to$) into the analogous fuzzy operators ($\widetilde{\neg}$, $\widetilde{\wedge}$, $\widetilde{\vee}$, $\widetilde{\to}$). For a reasonable QFM we should expect that the induced operators were fuzzy valid operators.

For a DFS the next property is guaranteed[6]:

Definition 13 (Property of the induced truth functions) *Truth operations induced by a quantifier fuzzification mechanism must be coherent with fuzzy logic; i.e., the following must hold:*
a. $\widetilde{id_{\mathbf{I}}} = \widetilde{F}(id_2)$ (where $id_2 : \mathbf{2} \to \mathbf{2}$ is the bivalued identity truth function) is the fuzzy identity truth function.
b. $\widetilde{\neg} = \widetilde{F}(\neg)$ is a strong negation operator.
c. $\widetilde{\wedge} = \widetilde{F}(\wedge)$ is a tnorm.
d. $\widetilde{\vee} = \widetilde{F}(\vee)$ is a tconorm.
e. $\widetilde{\to} = \widetilde{F}(\to)$ is an implication function.

In this manner it is guaranteed that the fuzzy operators that are generated are reasonable from the perspective of fuzzy logic. For example, for $\mathcal{F}(\mathbf{some})(tall, blond)$ where $tall = \{0.7/John\}$ and $blond = \{0.4/John\}$ it is guaranteed we obtaine the result of using the induced *tconorm* on $(0.7, 0.4)$.

[6] This is a resume of the longer exposition maked in [34, section 4.3].

3.1.4 External negation property (P.4)

Now we are going to present a set of three very important properties from a linguistic point of view. The properties of *external negation*, *internal negation* and *duality*. We will begin defining the external negation property [34, section 3.5]:

Definition 14 (External negation) *[34, pag. 93]The external negation of a semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \to \mathbf{I}$ *is defined by* $(\widetilde{\neg} Q)(Y_1, \ldots, Y_n) = \widetilde{\neg}(Q(Y_1, \ldots, Y_n))$ *for all* $Y_1, \ldots, Y_n \in \mathcal{P}(E)$*. The definition of* $\widetilde{\neg}\widetilde{Q} : \widetilde{\mathcal{P}}(E) \to \mathbf{I}$ *in the case of fuzzy quantifiers* $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \to \mathbf{I}$ *is analogous*[7]*.*

From a linguistic point of view, the external negation of *"all the students are spanish"* is *"not all the students are spanish"*.

A QFM correctly generalizes the external negation property if it fulfills the next property:[8]

Definition 15 (External negation property.) *[28, pag. 22], [34, section 3.5] Let* $Q : \mathcal{P}(E)^n \to \mathbf{I}$ *a semi-fuzzy quantifier.* $\mathcal{F}$ *fulfills the property of external negation if* $\mathcal{F}(\widetilde{\neg} Q) = \widetilde{\neg}\mathcal{F}(Q)$*.*

For example, the fulfillment of this property assures:

$$\mathcal{F}(\textbf{at most 10})(X_1, X_2) = \mathcal{F}(\widetilde{\neg}\textbf{at least 11})(X_1, X_2) = \widetilde{\neg}\mathcal{F}(\textbf{at least 11})(X_1, X_2)$$

That is, the equivalence between the expressions *"at most ten rich students are intelligent"* and *"no more than eleven rich students are intelligent"* is assured in the fuzzy case.

3.1.5 Internal negation property (P.5)

The internal negation or antonym of a semi-fuzzy quantifier is defined as:

Definition 16 (Internal negation.) *[34, pag. 93] Let a semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \to \mathbf{I}$ *of arity* $n > 0$ *be given. The internal negation* $Q\neg : \mathcal{P}(E)^n \to \mathbf{I}$ *of* Q *is defined by*

$$Q\neg(Y_1, \ldots, Y_n) = Q\neg(Y_1, \ldots, \neg Y_n)$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$*. The internal negation* $\widetilde{Q}\widetilde{\neg} : \widetilde{\mathcal{P}}(E)^n \to \mathbf{I}$ *of a fuzzy quantifier* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \to \mathbf{I}$ *is defined analogously, based on the given fuzzy complement* $\widetilde{\neg}$*.*

[7] The reasonable choice of the fuzzy negation $\widetilde{\neg} : \mathbf{I} \to \mathbf{I}$ is the induced negation of the QFM.

[8] The property of external negation is one of the initial axioms of the axiomatic framework presented in [28, pag. 22] to define the DFSs.

For example, the internal negation of **all** :$\mathcal{P}(E)^2 \to \mathbf{I}$ is **no** : $\mathcal{P}(E)^2 \to \mathbf{I}$ because

$$\mathbf{all}(Y_1, Y_2)\neg = \mathbf{all}(Y_1, \neg Y_2) = \mathbf{no}(Y_1, Y_2)$$

The definition of the property of internal negation is:[9]

Definition 17 (Internal negation property) *[28, pag. 22][34, section 3.5] Let $Q : \widetilde{\mathcal{P}}(E)^n \to \mathbf{I}$ be a semi-fuzzy quantifier of arity $n > 0$. A QFM $\mathcal{F}$ fulfills the property of internal negation if $\mathcal{F}(Q\neg) = \mathcal{F}(Q)\widetilde{\neg}$.*

For example, this property assures

$$\mathcal{F}(\mathbf{all})(X_1, X_2) = \mathcal{F}(\mathbf{all}\neg)(X_1, \widetilde{\neg}X_2) = \mathcal{F}(\mathbf{no})(X_1, \widetilde{\neg}X_2)$$

That is, the equivalence between the expressions *"all big houses are overvaluated"* and *"no big houses are undervaluated"* is assured in the fuzzy case.

3.1.6 Duality property (P.6)

This property is a consequence of the fulfillment of the external and internal negation properties. In [34] is one of the axioms used to define the DFSs.

Definition 18 (Dual quantifier.) *[33, pag. 99]The dual $Q\widetilde{\square} : \mathcal{P}(E)^n \to \mathbf{I}$ of a semi-fuzzy quantifier $Q\widetilde{\square} : \mathcal{P}(E)^n \to \mathbf{I}$, $n > 0$ is defined by*

$$Q\widetilde{\square}(Y_1, \ldots, Y_n) = \widetilde{\neg}Q(Y_1, \ldots, \neg Y_n)$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. The dual $\widetilde{Q}\widetilde{\square} = \widetilde{\neg}\widetilde{Q}\widetilde{\neg}$ of a fuzzy quantifier $\widetilde{Q}$ is defined analogously.

For example, the dual of **all** :$\mathcal{P}(E)^2 \to \mathbf{I}$ is

$$\mathbf{all}\widetilde{\square}(Y_1, Y_2) = \widetilde{\neg}\mathbf{all}(Y_1, \neg Y_2) = \mathbf{some}(Y_1, Y_2)$$

Using the axiom of duality [34, pag. 94-96] the duality property can be defined:

Definition 19 (Duality property) *We say that a QFM $\mathcal{F}$ fulfills the property of duality if for all semi-fuzzy quantifiers $Q : \widetilde{\mathcal{P}}(E)^n \to \mathbf{I}$ of arity $n > 0$ $\mathcal{F}\left(Q\widetilde{\square}\right) = \mathcal{F}(Q)\widetilde{\square}$.*

For example this property assures that

$$= \mathcal{F}(\mathbf{all})\widetilde{\square}(X_1, X_2) = \mathcal{F}(\mathbf{some})(X_1, X_2)$$

that is, the equivalence of the sentences *"not all the expensives cars are not good"* and *"some expensive car is good"* is assured in the fuzzy case.

[9] The property of internal negation is one of the initial axioms of the axiomatic framework presented in [28, pag. 22] to define the DFSs.

3.1.7 Internal meets property (P.7)

In combination with negation properties, this property assures boolean combination of arguments are mapped to the fuzzy case.

First, we show the "union" and "intersection" quantifiers:

Definition 20 (Union quantifier) *[34, section 3.7] Let $Q : \mathcal{P}(E)^n \to \mathbf{I}$ be a semi-fuzzy quantifier, $n > 0$, be given. We define the fuzzy quantifier $Q\cup : \mathcal{P}(E)^{n+1} \to \mathbf{I}$ as*

$$Q \cup (Y_1, \ldots, Y_n, Y_{n+1}) = Q(Y_1, \ldots, Y_{n-1}, Y_n \cup Y_{n+1})$$

for all $Y_1, \ldots, Y_{n+1} \in \mathcal{P}(E)$. In the case of fuzzy quantifiers $\widetilde{Q}\widetilde{\cup}$ is defined analogously, based on a fuzzy definition of $\widetilde{\cup}$.

Definition 21 (Intersection quantifier) *Let $Q : \mathcal{P}(E)^n \to \mathbf{I}$ a semi-fuzzy quantifier, $n > 0$, be given. We define the semi-fuzzy quantifier $Q\cap : \mathcal{P}(E)^{n+1} \to \mathbf{I}$ as*

$$Q \cap (Y_1, \ldots, Y_n, Y_{n+1}) = Q(Y_1, \ldots, Y_{n-1}, Y_n \cap Y_{n+1})$$

for all $Y_1, \ldots, Y_{n+1} \in \mathcal{P}(E)$. In the case of fuzzy quantifiers $\widetilde{Q}\widetilde{\cap}$ is defined analogously, based on a fuzzy definition of $\widetilde{\cap}$.

Expressions like *"all* Y_1 are Y_2 or Y_2" where Y_1, Y_2, Y_3 are crisp can be evaluated by means of less arity quantifiers with these constructions:

$$\mathbf{all} \cup (Y_1, Y_2, Y_3) = \mathbf{all}(Y_1, Y_2 \cup Y_3)$$

The definition of the property is:

Definition 22 (Internal meets property) *[34, pag. 97] Let $Q : \mathcal{P}(E)^n \to \mathbf{I}$ a semi-fuzzy quantifier, $n > 0$. We will say a QFM $\mathcal{F}$ preserves the property of internal meets if:*

$$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\,\widetilde{\cup}$$
$$\mathcal{F}(Q\cap) = \mathcal{F}(Q)\,\widetilde{\cap}$$

As a consequence,

$$\begin{aligned}\mathcal{F}(\exists)\left(X_1 \widetilde{\cap} X_2\right) &= \mathcal{F}(\exists)\,\widetilde{\cap}\,(X_1, X_2)\\ &= \mathcal{F}(\exists\cap)(X_1, X_2)\\ &= \mathcal{F}(\mathbf{some})(X_1, X_2)\end{aligned}$$

3.1.8 Monotonicity in arguments property (P.8)

In this section we present the property of monotonicity in arguments. This property is one of the axioms used to define the DFSs.

Definition 23 (Monotonicity) *[34, pag. 98] A semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \rightarrow \mathbf{I}$ is said to be nondecreasing in its i-th argument, $i \in \{1, \ldots, n\}$ if*

$$Q(Y_1, \ldots, Y_i, \ldots, Y_n) \leq Q(Y_1, \ldots, Y_{i-1}, Y_i', Y_{i+1}, \ldots, Y_n)$$

whenever the involved arguments $Y_1, \ldots, Y_n, Y_i' \in \mathcal{P}(E)$ satisfy $Y_i \subseteq Y_i'$. Q is said to be nonincreasing in the i-th argument if under the same conditions, it always holds that

$$Q(Y_1, \ldots, Y_i, \ldots, Y_n) \geq Q(Y_1, \ldots, Y_{i-1}, Y_i', Y_{i+1}, \ldots, Y_n)$$

The corresponding definitions for fuzzy quantifiers $Q : \widetilde{\mathcal{P}}(E)^n \rightarrow \mathbf{I}$ are entirely analogous. In this case, the arguments range over $\widetilde{\mathcal{P}}(E)$, and '$\subseteq$' is the usual fuzzy inclusion relation ($X_1 \subseteq X_2$ if $\mu_{X_1}(e) \leq \mu_{X_2}(e)$ for all $e \in E$).

For example, the semi-fuzzy quantifier **some** : $\mathcal{P}(E)^2 \rightarrow \mathbf{I}$ is monotonic nondecreasing in both arguments.

The next property guarantees the extension of the monotonicity to fuzzy quantifiers:

Definition 24 (Monotonicity property) *[34, pag. 100]A QFM $\mathcal{F}$ is said to preserve monotonicity in the arguments if semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \rightarrow \mathbf{I}$ which are nondecreasing (nonincreasing) in their i-th argument $i \in \{1, \ldots, n\}$ are mapped to fuzzy quantifiers $\mathcal{F}$ which are also nondecreasing (nonincreasing) in their i-ih argument.*

For example, if a QFM $\mathcal{F}$ guarantees this property then $\mathcal{F}(\mathbf{some}) : \widetilde{\mathcal{P}}(E)^2 \rightarrow \mathbf{I}$ is monotonic non-decreasing in both arguments.

3.1.9 Monotonicity in quantifiers property (P.9)

The *property of monotonicity in quantifiers* is a very important consequence of the DFS axioms [28, 34]. Independently, this property has also been defined in [46, pag. 73],[16] for unary quantifiers with the name of *property of inclusion of quantifiers*.

This property establishes that if a semi-fuzzy quantifier Q is included in other semi-fuzzy quantifier Q' (i.e., the results of Q are smaller than the results of Q' for all the selections of crisp arguments $Y_1, \ldots, Y_n \in \mathcal{P}(E)$) then the fuzzy extension $\mathcal{F}(Q)$ is also included in $\mathcal{F}(Q')$.

Definition 25 (Monotonicity in the quantifiers) *[34, pag. 128] Suppose $Q, Q' : \mathcal{P}(E)^n \rightarrow \mathbf{I}$ are semi-fuzzy quantifiers. Let us write $Q \leq Q'$ if for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$, $Q(Y_1, \ldots, Y_n) \leq Q'(Y_1, \ldots, Y_n)$. On fuzzy quantifiers we define $\leq$ analogously, based on arguments in $\widetilde{\mathcal{P}}(E)$.*

For example, for the following semi-fuzzy quantifiers

$$Q(X_1, X_2) = \begin{cases} S_{0.5,0.7}\left(\frac{|X_1 \cap X_2|}{|X_1|}\right) & X_1 \neq \varnothing \\ 1 & X_1 = \varnothing \end{cases} \tag{4}$$

$$Q'(X_1, X_2) = \begin{cases} S_{0.3,0.5}\left(\frac{|X_1 \cap X_2|}{|X_1|}\right) & X_1 \neq \varnothing \\ 1 & X_1 = \varnothing \end{cases}$$

it holds that $Q \leq Q'$.

The next property is defined based on the Theorem 4.32 in [34, pag. 128].

Definition 26 (Property of monotonicity in quantifiers) *Suppose $\mathcal{F}$ is a QFM, and $Q, Q' : \mathcal{P}(E)^n \to \mathbf{I}$ are semi-fuzzy quantifiers. We say that $\mathcal{F}$ fulfills the property of monotonicity in quantifiers if and only if $\mathcal{F}(Q) \leq \mathcal{F}(Q')$.*

This property guarantees that $\mathcal{F}(Q) \leq \mathcal{F}(Q')$ for the semi-fuzzy quantifiers defined on the expression 4.

3.1.10 Property of functional application (P.10)

The *property of compatibility with functional application* forms part of the axioms that are used to define the DFSs [34]. This property requires that a QFM must be compatible with its induced extension principle.

Definition 27 (Extension of a function to sets) *Let us consider $\beta : E \to S$ function. Function $\widehat{\beta} : \mathcal{P}(E) \to \mathcal{P}(S)$ is defined in the following way: $\widehat{\beta}(Y) = \{\beta(e) : e \in Y\}$.*

The extension principle induced by a QFM is defined as:

Definition 28 (Induced extension principle) *[34, pág. 101] All QFM $\mathcal{F}$ induce an extension principle $\widehat{\mathcal{F}}$ that to each function $f : E \to E'$ (where $E, E' \neq \varnothing$) assigns a function $\widehat{\mathcal{F}}(f) : \widetilde{\mathcal{P}}(E) \to \widetilde{\mathcal{P}}(E')$ defined by $\mu_{\widehat{\mathcal{F}}(f)(X)}(e') = \mathcal{F}\left(\chi_{\widehat{f}(\cdot)}(e')\right)(X)$ for all $X \in \widetilde{\mathcal{P}}(E)$, $e' \in E'$.*

It should be noted that in $\chi_{\widehat{f}(\cdot)}(e')$ the function $\widehat{f} : \mathcal{P}(E) \to \mathcal{P}(E')$ is the extension to sets of the function f and then $\chi_{\widehat{f}(\cdot)}(e')$ is the characteristic function of this extension; that is, $\chi_{\widehat{f}(\cdot)}(e')$ is a semi-fuzzy quantifier that for a set $Y \in \mathcal{P}(E)$ returns 1 if $e' \in \widehat{f}(Y)$ and 0 in other case.

The property of compatibility with functional application is defined as:

Proposition 29 (Compatibility with functional application) *[34, Pág. 104] Let $\mathcal{F}$ a given QFM. We will say that $\mathcal{F}$ is compatible with its induced extension principle if $\mathcal{F}\left(Q \circ \underset{i=1}{\overset{n}{\times}} \widehat{f}_i\right) = \mathcal{F}(Q) \circ \underset{i=1}{\overset{n}{\times}} \widehat{\mathcal{F}}(f_i)$ or equivalently*

$$\mathcal{F}\left(Q \circ \underset{i=1}{\overset{n}{\times}} \widehat{\mathcal{F}}(f_i)\right)(X'_1, \ldots, X'_n) = \mathcal{F}(Q)\left(\widehat{\mathcal{F}}(f_1)(X'_1), \ldots, \widehat{\mathcal{F}}(f_n)(X'_n)\right)$$

is valid for all semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \to \mathbf{I}$ *and all the function* $f_1, \ldots, f_n : E' \to E$ *with domain* $E' \neq \varnothing$, $X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E')$.

That is, if a QFM $\mathcal{F}$ fulfills the property of functional application, the same results are obtained when we first apply the induced extension principle to the argument sets $X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E')$ and then we apply the quantifier $\mathcal{F}(Q)$, and when we first apply the semi-fuzzy quantifier $Q \circ \underset{i=1}{\overset{n}{\times}} \widehat{f}_i$ (that to the crisp sets $Y'_1, \ldots, Y'_n \in \mathcal{P}(E')$ apply the function $\underset{i=1}{\overset{n}{\times}} \widehat{f}_i$, and then evaluates $Q : \mathcal{P}^n(E) \to \mathbf{I}$), and then we apply $\mathcal{F}$ to compute the function $\mathcal{F}\left(Q \circ \underset{i=1}{\overset{n}{\times}} \widehat{f}_i\right)$ on $X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E')$.

This propery is very important in union with the rest of the axioms used to define the QFMs because all toghether assures the fulfillment of a very important and intuitive set of properties.

3.2 The DFS axiomatic framework

We now present the DFS axiomatic framework. In [34] the author dedicates the whole 4 chapter to describe the properties that are consequence of the axiomatic framework. For the sake of brevity, we have only described the set of properties we have consider more relevant. Other important properties the author describes in [34] are argument permutations (the QFMs are compatible with the trasposition of arguments), cylindrical extensions (that guaratees vacuous arguments are irrelevant), quantitativity (QFMs guarantees that quantitative semi-fuzzy quantifiers are mapped to quantitative fuzzy quantifiers), etc.

The framework the author sets out in [34, section 3.9] is a refinement of the original framework defined on [28, pag. 22] that it was composed by 9 interdependent axioms. The two frameworks are equivalent. We present now the definition of the DFS framework:

Definition 30 *A QFM $\mathcal{F}$ is called a determiner fuzzification scheme (DFS) if the following conditions are satisfied for all semi-fuzzy quantifiers* $Q : \mathcal{P}(E)^n \to \mathbf{I}$.

Correct generalisation	$\mathcal{U}(\mathcal{F}(Q)) = Q \quad \text{if } n \leq 1$	*(Z-1)*
Projection quantifiers	$\mathcal{F}(Q) = \widetilde{\pi}_e \quad \text{if } Q = \pi_e \text{ for some } e \in E$	*(Z-2)*
Dualisation	$\mathcal{F}\left(Q\widetilde{\square}\right) = \mathcal{F}(Q)\,\widetilde{\square} \quad n > 0$	*(Z-3)*
Internal joins	$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\,\widetilde{\cup} \quad n > 0$	*(Z-4)*
Preservation of monotonicity	*If Q is nonincreasing in the n-th arg, then $\mathcal{F}(Q)$ is nonincreasing in n-th arg, $n > 0$*	*(Z-5)*
Functional application	$\mathcal{F}\left(Q \circ \underset{i=1}{\overset{n}{\times}} \widehat{f}_i\right) = \mathcal{F}(Q) \circ \underset{i=1}{\overset{n}{\times}} \widehat{\mathcal{F}}(f_i)$ *where* $f_1, \ldots, f_n : E' \to E, E' \neq \varnothing$	*(Z-6)*

In the previous definition $\mathcal{U} : \left(\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \to \mathbf{I}\right) \to \left(Q : \mathcal{P}(E)^n \to \mathbf{I}\right)$ is *the underlying semi-fuzzy quantifier* [34, pag. 75]; that is, the semi-fuzzy quantifier

$Q : \mathcal{P}(E)^n \to \mathbf{I}$ defined as:

$$\mathcal{U}\left(\widetilde{Q}\right)(Y_1, \ldots, Y_n) = \widetilde{Q}(Y_1, \ldots, Y_n)$$

for all crisp $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. The axiom 1 is equivalent to the fulfillment of the correct generalization property in the unary case.

3.3 Some properties that are not a consequence of the DFS axioms

Now we will describe some adequacy properties that are not guaranteed by the DFS framework because they impose an excesive restriction on the class of plausible models. In [34, chapter 6] a detailed exposition considering these and other properties can be consulted.

3.3.1 Property of continuity in arguments (P.11)

Continuity properties are fundamental. Models that do not fulfil these properties generally will not be valid from a practical viewpoint. One reason is that it is impossible to avoid measure errors and, as a consequence, errors in data measures could cause completely different analysis. Other reason is that from a user viewpoint, it would be very difficult to understand why no significant differences produce different results. Continutiy is also necessary from an application view (for example, imagine we need to use fuzzy quantifiers in a control system).

In this section we will explain the continuity in arguments property [34, Section 6.2]. The definition of this property is based on the next metric to measure the difference between two pairs of fuzzy sets $(X_1, \ldots, X_n), (X'_1, \ldots, X'_n) \in \widetilde{\mathcal{P}}(E)$:

Definition 31 **($d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n))$)** *[34, pag. 162] For all base sets $E \neq \varnothing$ and all $n \in \mathbb{N}$ the metric $d : \widetilde{\mathcal{P}}(E)^n \times \widetilde{\mathcal{P}}(E)^n \to \mathbf{I}$ is defined by*

$$d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) = \max_{i=1}^{n} \sup\left\{\left|\mu_{X_i}(e) - \mu_{X'_i}(e) : e \in E\right|\right\}$$

for all $X_1, \ldots, X_n, X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E)$.

Using this metric the property of continuity in arguments is defined:

Definition 32 (Continuity in arguments property) *[34, pag. 163] We say that a QFM $\mathcal{F}$ is arg-continuous if and only if $\mathcal{F}$ maps all semi-fuzzy quantifiers to continuous fuzzy quantifiers $\mathcal{F}(Q)$; i.e. for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $d\left(\mathcal{F}(Q)(X_1, \ldots, X_n), \mathcal{F}(Q)\left(X'_1, \ldots, X'_n\right)\right) < \varepsilon$ for all $X'_1, \ldots, X'_n \in \mathcal{P}(E)$ with $d\left((X_1, \ldots, X_n), \left(X'_1, \ldots, X'_n\right)\right) < \delta$*

3.3.2 Property of continuity in quantifiers (P.12)

In the same way we require continuity on argument sets, we also require continuity in quantifiers. That is, we do not expect big differences in results when we modify slightly the quantifiers.

The distance between two semi-fuzzy quantifiers is defined as:

Definition 33 ($d(Q, Q')$) *[34, pag. 163] For all semi-fuzzy quantifiers $Q, Q' : \mathcal{P}(E)^n \to \mathbf{I}$ the distance between Q and Q' is defined as:*

$$d\left(Q, Q^{'}\right) = \sup\left\{\left|Q(Y_1, \ldots, Y_n) - Q^{'}(Y_1, \ldots, Y_n)\right| : Y_1, \ldots, Y_n \in \mathcal{P}(E)^n\right\}$$

and similarity for all fuzzy quantifiers $\widetilde{Q}, \widetilde{Q'} : \widetilde{\mathcal{P}}(E)^n \to \mathbf{I}$

$$d\left(\widetilde{Q}, \widetilde{Q'}\right) = \sup\left\{\left|\mathcal{F}(Q)(X_1, \ldots, X_n) - \mathcal{F}\left(Q^{'}\right)(X_1, \ldots, X_n)\right| : X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)\right\}$$

Q-continuity is defined as:

Definition 34 (Continuity in quantifiers property) *[34, pag. 163] We say that a QFM $\mathcal{F}$ is Q-continuous if and only if for each semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \to \mathbf{I}$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that $d\left(\mathcal{F}(Q), \mathcal{F}\left(Q^{'}\right)\right) < \varepsilon$ whenever $Q^{'} : \mathcal{P}(E)^n \to \mathbf{I}$ satisfies $d\left(Q, Q^{'}\right) < \delta$.*

3.3.3 Property of the fuzzy argument insertion (P.13)

The property of fuzzy argument insertion is the fuzzy generalization of the crisp argument insertion [34, section 4.10]. Let $Q : \mathcal{P}^n(E) \to \mathbf{I}$ a semi-fuzzy quantifier $n > 0$, and $A \in \mathcal{P}(E)$. By $Q \lhd A : \mathcal{P}^{n-1}(E) \to \mathbf{I}$ we will denote the semi-fuzzy quantifier defined as

$$Q \lhd A(Y_1, \ldots, Y_{n-1}) = Q(Y_1, \ldots, Y_{n-1}, A)$$

for all $Y_1, \ldots, Y_{n-1} \in \mathcal{P}(E)$. As a consequence of the DFS axioms it is fulfilled that

$$\mathcal{F}(Q \lhd A) = \mathcal{F}(Q) \lhd A$$

for all semi-fuzzy quantifier Q of arity $n > 0$, and all crisp $A \in \mathcal{P}(E)$.

Fuzzy argument insertion cannot be modeled directly, because a semi-fuzzy quantifier $Q : \mathcal{P}^n(E) \to \mathbf{I}$, $n > 0$ only accepts crisp arguments; that is, for all $A \in \widetilde{\mathcal{P}}(E)$ fuzzy only $\mathcal{F}(Q) \lhd A$ is defined and no $Q \lhd A$. But as is explained in [34, sección 6.8], a QFM $\mathcal{F}$ and a semi-fuzzy quantifier $Q : \mathcal{P}^n(E) \to \mathbf{I}$ we can study if there exists a semi-fuzzy quantifier $Q' : \mathcal{P}^{n-1}(E) \to \mathbf{I}$ fulfilling

$$\mathcal{F}(Q) \lhd A = \mathcal{F}(Q') \qquad (5)$$

for all $A \in \widetilde{\mathcal{P}}(E)$.

The reasonable election Q' is the following:

Definition 35 *[34, pag. 172]Let $\mathcal{F}$ a QFM, $Q : \mathcal{P}(E)^{n+1} \to \mathbf{I}$ a semi-fuzzy quantifier and $A \in \widetilde{\mathcal{P}}(E)$ a fuzzy set. Then $Q\widetilde{\triangleleft}A : \mathcal{P}(E)^n \to \mathbf{I}$ is defined as*

$$Q\widetilde{\triangleleft}A = \mathcal{U}(\mathcal{F}(Q) \triangleleft A)$$

that is, $Q\widetilde{\triangleleft}A(Y_1, \ldots, Y_n) = \mathcal{F}(Q)(Y_1, \ldots, Y_n, A)$ for all crisp sets $Y_1, \ldots, Y_n \in \mathcal{P}(E)$.

In [34, sección 6.8] the author mentions $Q' = Q\widetilde{\triangleleft}A$ is the unique election fo Q' that could satisfy 5. It should be noted that if Q' satisfies $\mathcal{F}(Q) \triangleleft A = \mathcal{F}(Q')$ then also satisfies

$$Q' = \mathcal{U}(\mathcal{F}(Q')) = \mathcal{U}(\mathcal{F}(Q) \triangleleft A) = Q\widetilde{\triangleleft}A$$

The next property resumes the fulfillent of the fuzzy argument insertion in the fuzzy case:

Definition 36 *[34, pag. 172]Let $\mathcal{F}$ be a QFM. We will say $\mathcal{F}$ fulfills fuzzy argument insertion if for all semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \to \mathbf{I}$ of artity $n > 0$ and all $A \in \widetilde{\mathcal{P}}(E)$ fuzzy is fulfilled*

$$\mathcal{F}(Q) \triangleleft A = \mathcal{F}\left(Q\widetilde{\triangleleft}A\right)$$

This property has a very strong relation with *nested quantification*. Althoug the sufficiency of this property for a DFS to adequate model nested quantifiers, in [34, section 12.6] the author has state the necessity of fulfilling this property. Moreover, the fulfillment of this property for standard DFSs is only achieved by the $\mathcal{M}_{CX}$, a paradigmatic example of good theoretical behavior.

3.4 Some probabilistic properties

Now, we will present two properties of probabilistic nature that are fulfilled by a number of probabilistic models [16, 21, 17].

3.4.1 Property of averaging for the identity quantifier (P.14)

The fulfillment of this property for a QFM $\mathcal{F}$ assures that when we apply the model to the unary semi-fuzzy quantifier **identity**: $\mathcal{P}(E) \to \mathbf{I}$ we obtain the average of the membership grades. First of all, the definition of this semi-fuzzy quantifer is:

Definition 37 *The unary semi-fuzzy quantifier **identity**: $\mathcal{P}(E) \to \mathbf{I}$ is defined as*

$$\mathbf{identity}(Y) = \frac{|Y|}{|E|}, Y \in \mathcal{P}(E)$$

It should be noted that for the **identity** semi-fuzzy quantifier the addition of one element improves the result in $\frac{1}{m}$. That is, the improvement obtained with the addition of elements to the argument set is linear. We can interpret the meaning of this semi-fuzzy quantifier as “as many as possible”.

The definition of the property is:

Definition 38 (Property of averaging for the identity quantifier) *We will say that a QFM $\mathcal{F}$ fulfills the property of averaging for the identity quantifier if:*

$$\mathcal{F}(\mathbf{identity})(X) = \frac{1}{m}\sum_{j=1}^{m} \mu_X(e_j)$$

As a result of the fulfillment of the property of averaging for the identity quantifier, the improvement obtained in $\mathcal{F}^A(\mathbf{identity})(X)$ is linear with respect to the increase of the membership grades of the argument fuzzy set.

3.4.2 Property of the probabilistic interpretation of quantifiers (P.15)

Let us suppose we use a set of semi-fuzzy quantifiers (*"at most about 20%"*, *"about between 20% and 80%"*, *"at least about 80%"*) to split the quantification universe. Then, if semi-fuzzy quantifiers can be interpreted in a probabilistic way, the fulfillment of this property guarantees that fuzzy quantifiers also can be interpreted in a probabilistic way.

Definition 39 *We will say that a set of semi-fuzzy quantifiers $Q_1, \ldots, Q_r : \mathcal{P}^n(E) \to \mathbf{I}$ forms a probabilistic Ruspini partition of the quantification universe if for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$ it holds that*

$$Q_1(Y_1, \ldots, Y_n) + \ldots + Q_r(Y_1, \ldots, Y_n) = 1$$

Example 40 *The next set of quantifiers forms a probabilistic Ruspini partition of the quantification universe:*

$$\textbf{at most about } 20\%(Y_1,Y_2) = \begin{cases} T_{-\infty,0,0.2,0.4}\left(\frac{|Y_1\cap Y_2|}{|Y_1|}\right) & Y_1 \neq \varnothing \\ \frac{1}{3} & Y_1 = \varnothing \end{cases} \tag{6}$$

$$\textbf{about between } 20\% \textbf{ and } 80\%(Y_1,Y_2) = \begin{cases} T_{0.2,0.4,0.6,0.8}\left(\frac{|Y_1\cap Y_2|}{|Y_1|}\right) & Y_1 \neq \varnothing \\ \frac{1}{3} & Y_1 = \varnothing \end{cases}$$

$$\textbf{at least about } 80\%(Y_1,Y_2) = \begin{cases} T_{0.6,0.8,1,\infty}\left(\frac{|Y_1\cap Y_2|}{|Y_1|}\right) & Y_1 \neq \varnothing \\ \frac{1}{3} & Y_1 = \varnothing \end{cases}$$

because

$$\textbf{at most about } 20\%(Y_1,Y_2) + \textbf{about between } 20\% \textbf{ and } 80\%(Y_1,Y_2) +$$

$$\textbf{at least about } 80\%(Y_1,Y_2) = 1$$

for all $Y_1, Y_2 \mathcal{P}(E)$.

Definition 41 (Property of probabilistic interpretation of quantifiers) *We will say that a QFM $\mathcal{F}$ fulills the property of probabilistic interpretation of quantifiers if for all probabilistic Ruspini partitions of the quantification universe* $Q_1, \ldots, Q_r : \mathcal{P}(E)^n \to \mathbf{I}$ *it holds that*

$$\mathcal{F}(Q_1)(X_1, \ldots, X_n) + \ldots + \mathcal{F}(Q_r)(X_1, \ldots, X_n) = 1$$

This property is very interesting because let us to interpret the result of evaluating a fuzzy expression as a probability distribution on the labels related to the quantifiers.[10]

4 Probabilistic interpretation of fuzzy sets based on likelihood functions

In this section we use the interpretation of fuzzy sets based on likelihood functions to establish the necessary background to define the $\mathcal{F}^A$ model. In [21, 17] another probabilistic view of fuzzy sets have been used to define a probabilistic framework for the definition of QFMs, and some models in this framework have been presented.

The semantic interpretation of fuzzy sets based on likelihood functions [42, 47, 4, 26] interprets vagueness in the data as a consequence of making a random experiment in which a set of individuals are asked about the fulfillment of a certain property. Let us consider the following example:

Example 42 *To decide if the height value* 185*cm. is considered "tall for male adults" a random experiment is performed in which four individuals (henceforth voters) are asked about their opinion. Let us denote by P the statement "the value* 185*cm. is tall for male adults", by* $V = \{v_1, v_2, v_3, v_4\}$ *the set of voters and by* $C(v, P) \in \mathbf{2} = \{0, 1\}$*,* $v \in V$ *the answer for each voter. If*

$$C(v_1, P) = 1, C(v_2, P) = 0, C(v_3, P) = 1, C(v_4, P) = 1$$

then we can define the degree of fulfillment of the statement P as

$$\mu(P) = \frac{|v \in V : C(v, P) = 1|}{|V|} = \frac{3}{4}$$

The above experiment can be extended to the height values of the universe. Let be $h \in \mathbb{R}$. We can define the degree of fulfillment of the statement *"the value of height h is tall"* as:

$$\mu(\text{“}h \text{ is } tall\text{”}) = \Pr(\text{“}h \text{ is } tall\text{”}) = \frac{|v \in V : C(v, \text{“}h \text{ is } tall\text{”}) = 1|}{|V|}$$

[10] In [39] a probabilistic interpretation of quantifiers is also used under the label semantics interpretation of fuzzy sets.

In this way we can assign a degree of fulfillment to the reference universe. In the common notation of fuzzy sets we assign to the label *"tall"* the fuzzy set $tall \in \widetilde{\mathcal{P}}(\mathbb{R})$ defined as: $\mu_{tall}(h) = \mu(\text{“}h \text{ is } tall\text{”})$.

Under this view of fuzzy sets, $\mu_{tall}(h) > \mu_{tall}(h')$ indicates that is more probable that *"h is tall"* than *"h′ is tall"*.

One of the accepted suppositions of this view is to assume that the answer of one voter for a certain value $h \in \mathbb{R}$ does not constrain his answer for other element h'[11]. Let us suppose that the universe E is finite. As we are interpreting that $\mu_X(e) = \Pr(\text{“}e \text{ is } X\text{”})$ then under the independence assumption we have:

$$\Pr(\text{“}e \text{ is } X\text{”} \wedge \text{“}e' \text{ is } X\text{”}) = \Pr(\text{“}e \text{ is } X\text{”}) \cdot \Pr(\text{“}e' \text{ is } X\text{”}) = \mu_X(e) \cdot \mu_X(e')$$

We can apply the same idea to compute the probability that a crisp set $Y \in \mathcal{P}(E)$ was a representative of a fuzzy set $X \in \widetilde{\mathcal{P}}(E)$ when we suppose the base set E finite. The intuition is that this probability is the probability that only the elements in Y belongs to X:

Definition 43 **(**$\Pr(representative_X = Y)$**)** *Let $X \in \widetilde{\mathcal{P}}(E)$ be a fuzzy set, E finite. The probability of the crisp set $Y \in \mathcal{P}(E)$ to be a representative of the fuzzy set $X \in \widetilde{\mathcal{P}}(E)$ is defined as*

$$\Pr(representative_X = Y) = \prod_{e \in Y} \mu_X(e) \prod_{e \in E \setminus Y} (1 - \mu_X(e))$$

It should be pointed out that in the previous definition the probability points are the subsets of E. In this way the σ-algebra on which the probability is defined is $\mathcal{P}(E)$.

It is worthy to note that definition 43 can be explained without mention to probability theory. If we consider the product tnorm ($\wedge(x_1, x_2) = x_1 \cdot x_2$) and the Lukasiewicz implication then $\Pr(representative_X = Y)$ is the *equipotence* [1] between Y and X:

$$Eq(Y, X) = \wedge_{e \in E} (\mu_X(e) \rightarrow \mu_Y(e)) \wedge (\mu_Y(e) \rightarrow \mu_X(e))$$

If $e \in E$ then $\mu_Y(e) = 1$ and

$$\begin{aligned}(\mu_X(e) \rightarrow \mu_Y(e)) \wedge (\mu_Y(e) \rightarrow \mu_X(e)) &= (\mu_X(e) \rightarrow 1) \wedge (1 \rightarrow \mu_X(e)) \\ &= 1 \wedge (1 - 1 + \mu_X(e)) \\ &= \mu_X(e)\end{aligned}$$

If $e \notin E$ then $\mu_Y(e) = 0$ and

$$\begin{aligned}(\mu_X(e) \rightarrow \mu_Y(e)) \wedge (\mu_Y(e) \rightarrow \mu_X(e)) &= (\mu_X(e) \rightarrow 0) \wedge (0 \rightarrow \mu_X(e)) \\ &= (1 - \mu_X(e)) \wedge 1 \\ &= 1 - \mu_X(e)\end{aligned}$$

[11] The situation in which the answer of one voter for a value of the universe constrains its answer for other values is related to the interpretation of fuzzy sets based on random sets. This view is used in [21, 17] for proposing a probabilistic framework to define models of fuzzy quantification.

And then:

$$
\begin{aligned}
Eq(Y,X) &= (\wedge_{e\in E}\mu_X(e)) \wedge (\wedge_{e\notin E}(1-\mu_X(e))) \\
&= \prod_{e\in Y}\mu_X(e) \prod_{e\in E\setminus Y}(1-\mu_X(e))
\end{aligned}
$$

The next notation will be used for $\Pr(\text{representative}_X = Y)$ in the rest of the paper:

Notation 44 ($m_X(Y)$) *Let $X \in \widetilde{\mathcal{P}}(E)$ be a fuzzy set and $Y \in \mathcal{P}(E)$ a crisp set. We will denote $m_X(Y) = \Pr(\textit{representative}_X = Y)$.*

Let us see now an example in which this probability is computed:

Example 45 *Let be $E = \{e_1, e_2, e_3\}$ and $X \in \widetilde{\mathcal{P}}(E)$ the fuzzy set defined as: $X = \{0.8/e_1, 0.2/e_2, 0.6/e_3\}$. Then*

$$
\begin{aligned}
m_X(\{e_1,e_3\}) &= \prod_{e\in Y}\mu_X(e) \prod_{e\in E\setminus Y}(1-\mu_X(e)) \\
&= \mu_X(e_1)\times\mu_X(e_3)\times(1-\mu_X(e_2)) = 0.384
\end{aligned}
$$

Sometimes, we will need to restrict the probability of a fuzzy set $X^{E'} \in \widetilde{\mathcal{P}}(E)$ to a subset $E' \subseteq E$ of the referential. Let $X^{E'} \in \widetilde{\mathcal{P}}(E')$ be the projection of X in E'; that is, X' is the fuzzy set on E' defined as: $\mu_{X'}(e) = \mu_X(e), e \in E'$.

In this case, the probability of X' on E' is denoted as:

Notation 46 ($m_X^{E'}(Y)$) *Let $X \in \widetilde{\mathcal{P}}(E)$ be a fuzzy set and $E' \subseteq E$ a restriction of the base set, and $Y \in \mathcal{P}(E')$ a crisp set on E'. We will denote $m_X^{E'}(Y) = m_{X'}(Y)$ where X' is the projection of X on E'; that is, the fuzzy set defined as $\mu_{X'}(e) = \mu_X(e), e \in E'$.*

It should be noted that

$$
\begin{aligned}
\sum_{Y\in\mathcal{P}(E)\mid e\in Y} m_X(Y) &= \sum_{\{e\}\subseteq Y\subseteq E} m_X(Y) = \sum_{\{e\}\subseteq Y\subseteq E} \mu_X(e)\, m_X^{E\setminus\{e\}}(Y\setminus\{e\}) \\
&= \mu_X(e) \sum_{\varnothing\subseteq Y\subseteq E\setminus\{e\}} m_X^{E\setminus\{e\}}(Y) = \mu_X(e)
\end{aligned}
$$

We consider now the situation in which we want to compute the probability of two crisp sets $Y_1, Y_2 \in \mathcal{P}(E)$ to be respectively the representatives of the fuzzy sets $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. That is, we consider the computation of the probability of the event "$\textit{representative}_{X_1} = Y_1 \wedge \textit{representative}_{X_2} = Y_2$".

If the two fuzzy sets X_1, X_2 are related to different reference universes (i.e., intelligence and height) it is reasonable to suppose that the probability of Y_1

to be representative of X_1 is independent of the probability of Y_2 to be representative of ${X_2}$[12]. It should be noted that once we have assumed independency between the voter decisions for elements related to same property is natural to assume independency for elements related to different properties. Then we define:

Definition 47 ($\Pr(representative_{X_1} = Y_1 \wedge representative_{X_2} = Y_2)$) *Let $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ be fuzzy sets, $Y_1, Y_2 \in \widetilde{\mathcal{P}}(E)$ crisp sets, and E a finite referential. Under the independence assumption for properties the probability that Y_1 to be a representative of X_1 and Y_2 to be a representative of X_2 is:*

$$\Pr(representative_{X_1} = Y_1 \wedge representative_{X_2} = Y_2) = m_{X_1}(Y_1) \cdot m_{X_2}(Y_2)$$

where we have assumed that the sets X_1 and X_2 are based on independent concepts.

In the definition of the model $\mathcal{F}^A$ we assumme always the independence hypothesis. Even this could not seem appropiate in some cases [42, 47] [42, 47], this hypothesis simplify considerably the definition of the models, it allows us a relative straigforwardly algebraic manipulation, and the definition of efficient algorithms.

5 The QFM $\mathcal{F}^A$

In this section we define the finite QFM $\mathcal{F}^A$ [23, 22, 17]. This model is based on the probabilistic interpretation of fuzzy sets previously explained.

Using expressions 43 and 47 the definition of the QFM $\mathcal{F}^A$ is easily made:

Definition 48 ($\mathcal{F}^A$) *[23, pag. 1359]Let $Q : \mathcal{P}(E)^n \to \mathbf{I}$ be a semi-fuzzy quantifier, E finite. The QFM $\mathcal{F}^A$ is defined as*

$$\mathcal{F}^A(Q)(X_1, \ldots, X_n) = \sum_{Y_1 \in \mathcal{P}(E)} \ldots \sum_{Y_n \in \mathcal{P}(E)} m_{X_1}(Y_1) \ldots m_{X_n}(Y_n) Q(Y_1, \ldots, Y_n) \tag{7}$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

In expression 7 we are assuming that the probability of being Y_i a representative of the fuzzy set X_i is independent of the probability of being Y_j a representative of the fuzzy set X_j for $i \neq j$. $\mathcal{F}^A(Q)(X_1, \ldots, X_n)$ can be interpreted as the average opinion of voters.

The next expression is an alternative definition of the model $\mathcal{F}^A$:

$$\mathcal{F}^A(X_1, \ldots, X_n) = \bigvee_{Y_1 \in \mathcal{P}(E)} \ldots \bigvee_{Y_n \in \mathcal{P}(E)} Eq(Y_1, X_1) \wedge \ldots \wedge Eq(Y_n, X_n) \wedge Q(Y_1, \ldots, Y_n)$$

[12] In the work [47] is analyzed deeply the interpretation of fuzzy sets based on likelihood functions. In [47, Pág. 95] the author argue that the most reasonable is to suppose independence between different universes.

where $\vee$ the Lukasiewicz tconorm ($\vee(x_1, x_2) = \min(x_1 + x_2, 1)$), $\wedge$ is the product tnorm ($\wedge(x_1, x_2) = x_1 \cdot x_2$) and $Eq(Y, X)$ is the equipotence between the crisp set Y and the fuzzy set X, that it was defined in previous section. In this way, the model can be defined without mention to probability theory.

The following example shows the application of the QFM $\mathcal{F}^A$:

Example 49 *Let us consider the sentence*

"Nearly all the intelligent workers are well paid"

where the semi-fuzzy quantifier $Q =$ ***"nearly all"****, and the fuzzy sets* ***"intelligent workers"*** *and* ***"well paid"*** *take the following values:*

$$\begin{aligned}
\textbf{intelligent workers} &= \{0.8/e_1, 0.9/e_2, 1/e_3, 0.2/e_4\} \\
\textbf{well paid} &= \{1/e_1, 0.8/e_2, 0.3/e_3, 0.1/e_4\} \\
Q(X_1, X_2) &= \begin{cases} \max\left\{2\left(\frac{|X_1 \cap X_2|}{|X_1|}\right) - 1, 0\right\} & X_1 \neq \varnothing \\ 1 & X_1 = \varnothing \end{cases}
\end{aligned}$$

We compute the probabilities of the representatives of the fuzzy sets ***"intelligent workers"*** *and* ***"well paid"****:*

$$\begin{aligned}
m_{\textbf{intelligent workers}}(\varnothing) &= (1-0.8)(1-0.9)(1-1)(1-0.2) = 0 \\
m_{\textbf{intelligent workers}}(\{e_1\}) &= 0.8(1-0.9)(1-1)(1-0.2) = 0 \\
&\ldots \\
m_{\textbf{intelligent workers}}(\{e_1, e_2, e_3, e_4\}) &= 0.8 \cdot 0.9 \cdot 1 \cdot 0.2 = 0.144 \\
m_{\textbf{well paid}}(\varnothing) &= (1-1)(1-0.8)(1-0.3)(1-0.1) = 0 \\
m_{\textbf{well paid}}(\{e_1\}) &= 1 \cdot (1-0.8)(1-0.3)(1-0.1) = 0.126 \\
&\ldots \\
m_{\textbf{well paid}}(\{e_1, e_2, e_3, e_4\}) &= 0.8 \cdot 0.9 \cdot 1 \cdot 0.2 = 0.144
\end{aligned}$$

And using expression 7:

$$\begin{aligned}
&\mathcal{F}^A(Q)(\textbf{intelligent workers}, \textbf{well paid}) \\
&= \sum_{Y_1 \in \mathcal{P}(E)} \sum_{Y_2 \in \mathcal{P}(E)} m_{X_1}(Y_1)\, m_{X_2}(Y_2)\, Q(Y_1, Y_2) = 0.346
\end{aligned}$$

The QFM $\mathcal{F}^A$ fulfills the DFS axiomatic framework; that is, the QFM $\mathcal{F}^A$ is a finite DFS. Moreover, the QFM $\mathcal{F}^A$ fulfills the additional properties we have presented in this technical report[13]. The analysis of properties of the model is made in the appendix.

[13] With respect to the additional properties defined in [34, Chapter 6]. we mention that the $\mathcal{F}^A$ model does not fulfill the "propagation of fuzziness property" (basically because is not fulfilled by the induced operators). Other reasonable properties, as conservativity, are in contradiction with the DFS framework, and cannot be fulfilled.

6 Some additional results about the $\mathcal{F}^A$ model

6.1 Limit case approximation of the $\mathcal{F}^A$ model

In this section we will prove that the asintotic behavior of the $\mathcal{F}^A$ model for unary proportional quantifiers is the Zadeh's model. As a practical consequence, the $\mathcal{F}^A$ model can be approximated in linear time when the base set is composed of a large number of elements.

To prove this property we will use a particular result derived of the central limit theorem [12, page 263].

Theorem 50 ***Central limit theorem applied to Bernoulli variables***. *Let $X_1, \ldots, X_m$ be independent random variables, each X_i following a Bernoulli distribution with parameter p_i. Moreover, let us suppose that the infinite sum $\sum_{i=1}^{\infty} p_i (1 - p_i)$ is divergent and let Y_n be*

$$Y_m = \frac{\sum_{i=1}^{m} X_i - \sum_{i=1}^{n} p_i}{\left(\sum_{i=1}^{m} p_i q_i\right)^{1/2}}$$

Then

$$\lim_{n \to \infty} \Pr\left(Y_m \leq x\right) = \Phi\left(x\right)$$

where $\Phi(x)$ is the standard normal distribution function.

Using this result we will prove $\mathcal{F}^A$ approximation for proportional unary quantifiers.

Theorem 51 *Let $Q : \mathcal{P}(E^m) \to \mathbf{I}$ be a unary proportional semi-fuzzy quantifier defined by means of a continuous proportional fuzzy number $\mu_Q : [0,1] \to \mathbf{I}$*

$$Q(Y) = \mu_Q\left(\frac{|Y|}{|E|}\right)$$

for $Y \in \mathcal{P}(E)$. Let $e_{1,\ldots,}e_m$ a succession and $X_m \in \widetilde{\mathcal{P}}(E)$ be a fuzzy set constructed on such sucession. Let $Y_m = \{\mu_X(e) : \mu_X(e) = 1 \vee \mu_X(e) = 0\}$ the crisp set constructed with the crisp elements of X. If the following limit there exists:

$$\lim_{m \to \infty} \frac{|Y_m|}{|E_m|}$$

Then, when the size of the base set E tends to infinite, $\mathcal{F}^A(Q)(X)$ tends to:

$$\lim_{|E| \to \infty} \mathcal{F}^A(Q)(X) = fn\left(\frac{\sum_{e \in E} \mu_X(e)}{|E|}\right)$$

for $X \in \widetilde{\mathcal{P}}(E)$.

The assumption of existence of the limit $\lim_{m\to\infty}\frac{|Y_m|}{|E_m|}$ is very weak and irrelevant from a practical point of view. We simply are asking the proportional cardinality of the succession Y_m does not oscillate as m tends to infinite.

Proof. The interpretation underlying the $\mathcal{F}^A$ model assumes that each $e_i \in E$ represents and independent Bernoulli process of probability $\mu_X(e_i)$. Let us denote X_i this Bernoulli process. ■

The probability

$$\Pr(card_X = i) = \sum_{Y\in\mathcal{P}(E)} m_X(Y)$$

represents the probability of the random variable $X_m = \sum_{i=1}^{|E|} X_i$.

First, let us suppose that $\lim_{|E|\to\infty}\mu_X(e_i)(1-\mu_X(e_i))$ is divergent. Using previous theorem, for an enough big $m = |E|$, we can approximate this distribution for a normal distribution of parameters $\overline{X_m} = \sum_{i=1}^{m}\mu_X(e_i)$ and $\sigma_m^2 = \sum_{i=1}^{m}\mu_X(e_i)(1-\mu_X(e_i))$[14].

As the fuzzy number associated to the fuzzy quantifier is defined on $[0,1]$, we can use the transformation $Y = \frac{X_m}{m}$ to adapt this distribution to the $[0,1]$ interval. The probability distribution of Y is a normal distribution of parameters $\left(\frac{\mu}{m},\frac{\sigma^2}{m^2}\right)$.

Let us note that the normal distribution fulfills that the probability in k standard deviations of the mean is identical for all the normal distributions. As k tends to infinite, the probability mass in $(\mu - k\sigma, \mu + k\sigma)$ tends to 1. That is, we always can find a k such that the probability mass in $(\mu - k\sigma, \mu + k\sigma)$ would be as close to 1 as we wanted.

Let us compute the limit of the variance when m tends to infinite

$$\lim_{m\to\infty}\frac{\sigma^2}{m^2} = \lim_{m\to\infty}\frac{\sum_{i=1}^{m}\mu_X(e_i)(1-\mu_X(e_i))}{m^2} \leq \lim_{m\to\infty}\frac{\sum_{i=1}^{m}1}{m^2} = \lim_{m\to\infty}\frac{m}{m^2} = 0$$

That is, the probability distribution of Y is more and more concentrated as m tends to infinite. And then, for every $\delta,\theta > 0$ we can find a sufficiently large m such that $\Pr\left(\frac{\mu}{m}-\delta, \frac{\mu}{m}+\delta\right) > 1-\theta$.

As the fuzzy number μ_Q is continuous, then it is clear that

$$\lim_{|E|\to\infty}\mathcal{F}^A(Q)(X) = fn\left(\frac{\sum_{e\in E}\mu_X(e)}{|E|}\right)$$

when $m = |E|$ tends to infinite.

Let us consider now that $\sum_{i=1}^{\infty} p_i(1-p_i)$ is finite. In this case, we cannot apply the central limit theorem.

In this situation there are an infinite number of is such that $p_i = 1$ or $p_i = 0$. Moreover, when m tends to infinite, the proportion of is such that $p_i q_i \neq 0$ with respect to m tends to 0:

[14] Let X be a random variable following a normal distribution of parameters (μ,σ). If we define $Y = aX + b$ then Y follows a normal distribution of parameters $\left(a\mu + b, a^2\sigma^2\right)$.

$$\lim_{|E|\to\infty} \frac{|e : \mu_X(e) \neq 1 \wedge \mu_X(e) \neq 0|}{|E|} = 0$$

Then, X tends to a set in which only a finite number of elements are fuzzy. Let $Y = \{\mu_X(e) : \mu_X(e) = 1 \vee \mu_X(e) = 0\}$ and let k the finite numbers of is such that $p_i q_i \neq 0$.

It should be noted that by supposition the following limit there exists:

$$\lim_{|E|\to\infty} \frac{|Y|}{|E|} = c$$

For a sufficiently large m, all is such that $p_i q_i \neq 0$ are in X_m. Let us consider the "shape" of the probability distribution of X_m. As the is such that $p_i q_i = 0$ are crisp, the probability distribution of X_m will consist on $k+1$ points with $\Pr(i) \neq 0$ and $m-k-1$ points with $\Pr(i) = 0$. Moreover, by construction of $\Pr(i)$ all these points are consecutive.

When we normalize X_m to apply the fuzzy number that define the quantifier (by means ot the transformation $Y = \frac{X_m}{m}$) we are in the same case that when $\sum_{i=1}^{\infty} p_i(1-p_i)$ is divergent. The probability distribution will be more and more concentrated around $\frac{\mu}{m}$ and the approximation would be valid.

Although we have not developed a similar approximation for other kinds of quantifiers, it seems easy to extend previous proof to more complex cases. For example, in the case of proportional quantifiers we will have to consider two Bernoulli successions $X_{(1)m} = \mu_{X_1}(e_1), \ldots, \mu_{X_1}(e_m)$ and $X_{(2)m} = \mu_{X_2}(e_1), \ldots, \mu_{X_2}(e_m)$. As we are assuming independence, we can built a third Bernoulli succession $Z_m = \mu_{X_1}(e_1)\mu_{X_2}(e_1), \ldots, \mu_{X_1}(e_m)\mu_{X_2}(e_m)$ and the previous results can be applied for approximating the probability distribution of the cardinality of $X_1 \widetilde{\cap} X)$. When m tends to infinite, the probability distribution of Z_m tends to $\left(\frac{\sum \mu_{X_1 \tilde{\cap} X_2}(e_i)}{m}, 0\right)$. For the same reason, the probability distribution of X_1 tends to $\left(\frac{\sum \mu_{X_1}(e_i)}{m}, 0\right)$. And then, the proportional cardinality of X_2 in X_1 tends to $\frac{\sum \mu_{X_1 \tilde{\cap} X_2}(e_i)}{\sum \mu_{X_1}(e_i)}$.

6.2 Applying the $\mathcal{F}^A$ model to continuous fuzzy signals: Temporal Quantification.

The limit case approximation of the $\mathcal{F}^A$ opens the possibility of applying the model to continuous fuzzy signals, fundamental for the application of the model for *fuzzy quantified temporal reasoning* [11, 10, 44, 43].

Let us consider a continuous fuzzy signal[15] $S(t)$ where t represents time in an interval $E = [t_0, t_1]$. And let us suppose we want to evaluate a proportional

[15] The same argument allow us to apply the model to a non continuous signal with at most, a finite number of discontinuities. From a practical point of view, this is enough for applications.

quantifier $Q : \mathcal{P}([t_0, t_1]) \rightarrow \mathbf{I}$ on S where Q defined by means of a continuous fuzzy number. For example, Q could be defined as:

$$Q(Y) = S_{0.6,0.8}\left(\frac{\lambda(Y)}{\lambda(E)}\right)$$

where λ represents the Lebesgue measure.

As the $\mathcal{F}^A$ model is finite, it cannot be directly applied to continuous quantifiers. A reasonable possibility to apply the $\mathcal{F}^A$ model on a continuous set is to discretize the interval $[t_0, t_1]$ in m subintervals, $h = \frac{t_1 - t_0}{m}$ and to compute the result of the model in $E = \{e_0 = t_0, e_1 = t_0 + h, \ldots, e_m = t_1\}$. It should be noted that in the crisp case, as m tends to infinite, $Q(\{\chi_Y(x_0), \ldots, \chi_Y(x_m)\})$ tends to $Q(Y)$.

Let us consider the behavior of this approach in the limit case. Let $X \in \widetilde{\mathcal{P}}(E)$ be the fuzzy set defined as $\mu_X(x_i) = S(x_i)$

$$\lim_{m\rightarrow\infty} \mathcal{F}^A(Q)(X) = \lim_{m\rightarrow\infty} \sum_{Y\in\mathcal{P}(E)} m_X(Y)\,Q(Y)$$

and by using the limit approximation of the $\mathcal{F}^A$ model:

$$\begin{aligned}
\lim_{m\rightarrow\infty} \mathcal{F}^A(Q)(X) &\approx \lim_{m\rightarrow\infty} \mu_Q\left(\frac{\sum_{e_i} \mu_X(e_i)}{m}\right) \\
&= \lim_{m\rightarrow\infty} \mu_Q\left(\frac{h}{h}\frac{\sum_{e_i} \mu_X(e_i)}{m}\right) \\
&= \lim_{m\rightarrow\infty} \mu_Q\left(\frac{\sum_{e_i} h\mu_X(e_i)}{hm}\right) \\
&= \lim_{m\rightarrow\infty} \mu_Q\left(\frac{\sum_{e_i} h\mu_X(e_i)}{t_1 - t_0}\right)
\end{aligned}$$

when m tends to infinite, $\sum_{e_i} h\mu_X(e_i)$ tends to $\int S(t)\,dt$ (as $\mu_X(e_i) \in [\inf\{S(x) : e_i \leq x < e_i + h\}, \sup\{S(x) : e_i \leq x < e_i + h\}]$ then $\sum_{e_i} h\mu_X(e_i)$ is between the inferior integral and the superior integral of S). And then,

$$\lim_{m\rightarrow\infty} \mathcal{F}^A(Q)(X) = \mu_Q\left(\frac{\int S(t)\,dt}{t_1 - t_0}\right)$$

6.3 Applying the $\mathcal{F}^A$ model to a population described by means of a probability distribution

Let f be a probability distribution and *label* a fuzzy label defined on the referential universe of f. For example, f could be a normal distribution of parameters (μ, σ) representing the probability of *"heights for male adults"*, and *label* the fuzzy label *"being tall"*.

Let $Z = Z_1, \ldots, Z_m$ a random sample of f, and let $X = \{ \mu_X(z_i) = label(z_i)/z_i : i = 1, \ldots, m\}$. Then

$$\lim_{m\to\infty} \mathcal{F}^A(Q)(X) = \lim_{m\to\infty} \sum_{Y\in\mathcal{P}(E)} m_X(Y)\, Q(Y)$$

and by using the limit approximation of the $\mathcal{F}^A$ model:

$$\begin{aligned}
\lim_{m\to\infty} \mathcal{F}^A(Q)(X) &= ... \\
&= f_Q\left(\lim_{m\to\infty} \frac{\sum_i \mu_X(z_i)}{m}\right) \\
&= f_Q\left(\lim_{m\to\infty} \frac{1}{m}\sum_i \mu_X(z_i)\right) \\
&= f_Q\left(\overline{Z_i}\right) \\
&= f_Q\left(\int p(x)\, \mu_X(x)\, dx\right)
\end{aligned}$$

that is, the probability of the fuzzy event [57], or the probability of the label given the population distribution.

7 Conclusions

In this report we described and studied the theoretical behavior of the $\mathcal{F}^A$ QFM[16]. The analysis have proved the model is a finite DFS [34] essentially different of the standard DFSs proposed by this author. Moreover the underlying probabilistic semantics makes the model particularly interesting for applications.

Other interesting results are the limit case approximation of the model, that allows its application to continuous domains, and the study of the application of the model to populations described by means of a probability distributions.

8 Apendix A. Analysis of properties of the $\mathcal{F}^A$ QFM

In this section we analyze the most relevant properties of the QFM $\mathcal{F}^A$. A sligthly more detailed discussion can be consulted in [17].

First of all, we will proof some preliminary results.

Lemma 52 *It holds that*
1) $\widetilde{\mathcal{F}^A}(id_2)(x) = \widetilde{id}_{\mathbf{I}}(x)$ *where* $id_2 : \mathbf{2} \to \mathbf{2}$ *is the bivalued identity and* $\widetilde{id}_{\mathbf{I}} : \mathbf{I} \to \mathbf{I}$ *is the fuzzy identity.*

[16] Most of the theoretical analysis have been previously published in [17], in spanish.

2) $\widetilde{\mathcal{F}^A}(\neg) = \widetilde{\neg}(x) =$ *where* $\widetilde{\neg}$ *is the standard negation.*
3) $\widetilde{\mathcal{F}^A}(\wedge)(x_1, x_2) = x_1 \times x_2$*; that is, the product tnorm.*
4) $\widetilde{\mathcal{F}^A}(\vee)(x_1, x_2) = \widetilde{\neg}\widetilde{\mathcal{F}^A}(\wedge)(\widetilde{\neg}x_1, \widetilde{\neg}x_2) = x_1 + x_2 - x_1 \cdot x_2$*; that is, the probabilistic tconorm, the dual of the product.*
5)$\widetilde{\mathcal{F}^A}(\rightarrow)(x_1, x_2) = 1 - x_1 + x_1 \cdot x_2$*, in this case the Rechenbach fuzzy implication.*

Proof. We only are going to show the proof of $\widetilde{\mathcal{F}^A}(\vee)(x_1, x_2) = x_1 + x_2 - x_1 \times x_2$. The rest of the proofs can be consulted in [17, appendix A].

First, note that the definición of $Q_\vee : \mathcal{P}(\{1,2\}) \to \mathbf{I}$ is:

$$\begin{aligned} Q_\vee(\varnothing) &= \vee\left(\eta^{-1}(\varnothing)\right) = \vee(0,0) = 0 \\ Q_\vee(\{1\}) &= \vee\left(\eta^{-1}(\{1\})\right) = \vee(1,0) = 1 \\ Q_\vee(\{2\}) &= \vee\left(\eta^{-1}(\{2\})\right) = \vee(0,1) = 1 \\ Q_\vee(\{1,2\}) &= \wedge\left(\eta^{-1}(\{1,2\})\right) = \wedge(1,1) = 1 \end{aligned}$$

Then

$$\begin{aligned} \widetilde{\mathcal{F}^A}(x_1, x_2) &= \mathcal{F}^A(Q_\vee)\left(\eta^{-1}(x_1, x_2)\right) \\ &= \mathcal{F}^A(Q_\vee)(\{x_1/1, x_2/2\}) = \sum_{Y \in \mathcal{P}(\{1,2\})} m_{\{x_1/1, x_2/2\}}(Y)\, Q(Y) \\ &= m_{\{x_1/1,x_2/2\}}(\varnothing)\, Q(\varnothing) + m_{\{x_1/1,x_2/2\}}(\{1\})\, Q(\{1\}) \\ &+ m_{\{x_1/1,x_2/2\}}(\{2\})\, Q(\{2\}) + m_{\{x_1/1,x_2/2\}}(\{1,2\})\, Q(\{1,2\}) \\ &= \mu_{\{x_1/1,x_2/2\}}(1)\left(1 - \mu_{\{x_1/1,x_2/2\}}(2)\right) + \\ &\left(1 - \mu_{\{x_1/1,x_2/2\}}(1)\right) \mu_{\{x_1/1,x_2/2\}}(2) + \mu_{\{x_1/1,x_2/2\}}(1)\, \mu_{\{x_1/1,x_2/2\}}(2) \\ &= x_1(1 - x_2) + (1 - x_1)\, x_2 + x_1 x_2 \\ &= x_1 + x_2 - x_1 x_2 \end{aligned}$$

■

Moreover, in the proofs of the properties of the $\mathcal{F}^A$ model we need the following lemmas too.

Lemma 53 *Let* $X, Y \in \mathcal{P}(E)$ *be crisp sets. It holds that*

$$m_X(Y) = \begin{cases} 0 & : \quad X \neq Y \\ 1 & : \quad X = Y \end{cases}$$

Proof. The definition of $m_X(Y)$ is

$$m_X(Y) = \prod_{e \in Y} \mu_X(e) \prod_{e \in E \setminus Y} (1 - \mu_X(e))$$

and as X is crisp $\mu_X(e) = 1$ if $e \in E$ and $\mu_X(e) = 0$ if $e \notin E$. ■

Lemma 54 *Let $X \in \widetilde{\mathcal{P}}(E)$ be a fuzzy set $E' \subseteq E, E'' = E \backslash E'$ (that is $E' \cup E'' = E$). Let $Y \in \mathcal{P}(E)$ be a crisp set. Then*[17]*,*

$$m_X^E(Y) = m_X^{E'}\left(Y^{E'}\right) m_X^{E''}\left(Y^{E''}\right)$$

Proof. By definition of $m_X^E(Y)$, the probability of $m_X^E(Y)$ is the product of the probabilities on their projections:

$$\begin{aligned} m_X^E(Y) &= \prod_{e \in Y} \mu_X(e) \prod_{e \in E \backslash Y} (1 - \mu_X(e)) \\ &= \prod_{e \in Y \cap E'} \mu_X(e) \prod_{e \in (E \backslash Y) \cap E'} (1 - \mu_X(e)) \cdot \prod_{e \in Y \cap E''} \mu_X(e) \prod_{e \in (E \backslash Y) \cap E''} (1 - \mu_X(e)) \\ &= m_X^{E'}\left(Y^{E'}\right) m_X^{E''}\left(Y^{E''}\right) \end{aligned}$$

■

Lemma 55 *Let*

$$\widetilde{\vee}(x_1, x_2) = x_1 + x_2 - x_1 x_2$$

be the probabilistic tconorm. By $\widetilde{\vee}(x_1, \ldots, x_m)$ we denote its m-ary version; that is,

$$\widetilde{\vee}(x_1, \ldots, x_m) = \widetilde{\vee}\left(x_1, \widetilde{\vee}\left(x_2, \widetilde{\vee}\left(x_3 \ldots, \widetilde{\vee}(x_{m-1}, x_m)\right)\right)\right)$$

It is fulfilled[18]

$$\widetilde{\vee}(x_1, \ldots, x_m) = 1 - \prod_{i=1}^{m} (1 - x_i)$$

Proof. Proof is by induction.

Case $i = 2$:

$$\widetilde{\vee}(x_1, x_2) = x_1 + x_2 - x_1 x_2$$

and then

$$\begin{aligned} 1 - \prod_{i=1}^{2} (1 - x_i) &= 1 - (1 - x_1)(1 - x_2) \\ &= 1 - (1 - x_1 - x_2 + x_1 x_2) \\ &= x_1 + x_2 - x_1 x_2 \end{aligned}$$

[17] It should be remembered that with the notation $X^{E'}$ where $X \in \widetilde{\mathcal{P}}(E)$ is a fuzzy set we represent the restriction of X to the reference universe $E' \subseteq E$.

[18] It should be noted that for $\widetilde{\vee}(x) = x$ it also is fulfilled that $1 - \prod_{i=1}^{1} (1 - x) = 1 - (1 - x) = x$. Moreover, if we define $\widetilde{\vee}() = 0$ and $\prod_{i \in \varnothing} = 1$ previous relationship is also fulfilled.

Induction supposition: Case $i = m$

$$\widetilde{\vee}(x_1, \ldots, x_m) = 1 - \prod_{i=1}^{m} (1 - x_i)$$

Case $i = m + 1$

$$\begin{aligned}
\widetilde{\vee}(x_1, \ldots, x_{m+1}) &= \widetilde{\vee}\left(\widetilde{\vee}(x_1, \ldots, x_m), x_{m+1}\right) \\
&= \widetilde{\vee}\left(1 - \prod_{i=1}^{m} (1 - x_i), x_{m+1}\right) \\
&= 1 - \prod_{i=1}^{m} (1 - x_i) + x_{m+1} - \left(\left(1 - \prod_{i=1}^{m} (1 - x_i)\right) \times x_{m+1}\right) \\
&= \left(1 - \prod_{i=1}^{m} (1 - x_i)\right)(1 - x_{m+1}) + x_{m+1} \\
&= (1 - x_{m+1}) - (1 - x_{m+1}) \prod_{i=1}^{m} (1 - x_i) + x_{m+1} \\
&= (1 - x_{m+1}) - \prod_{i=1}^{m+1} (1 - x_i) + x_{m+1} \\
&= 1 - \prod_{i=1}^{m+1} (1 - x_i)
\end{aligned}$$

■

8.1 The $\mathcal{F}^A$ QFM is a DFS

Axiom Z-1 It holds that

$$\mathcal{U}\left(\mathcal{F}^A(Q)\right) = Q \quad \text{if } n \leq 1$$

Proof. Let $Q : \mathcal{P}(E) \to \mathbf{I}$ be a semi-fuzzy quantifier, $Y \in \mathcal{P}(E)$ a crisp set. Using the lemma 53 we have

$$\begin{aligned}
\mathcal{F}^A(Q)(Y) &= \sum_{Z \in \mathcal{P}(E)} m_Y(Z)\, Q(Z) = m_Y(Y)\, Q(Y) \\
&= Q(Y)
\end{aligned}$$

And then $\mathcal{U}(\mathcal{F}(Q))(Y) = Q(Y)$ for all $Y \in \mathcal{P}(E)$. ■

Axiom Z-2 It holds that

$$\mathcal{F}^A(Q) = \widetilde{\pi_e} \quad si\ Q = \pi_e \text{ for some } e \in E$$

Proof. Using the lemma 54

$$\begin{aligned}\mathcal{F}^A(\pi_e)(X) &= \sum_{Y\in\mathcal{P}(E)} m_X(Y)\,\pi_e(Y) = \sum_{Y\in\mathcal{P}(E)} m_X(Y)\,\chi_Y(e)\\ &= \sum_{Y\in\mathcal{P}(E)|e\in Y} m_X(Y) = \sum_{\{e\}\subseteq Y\subseteq E} m_X(Y)\\ &= \sum_{\{e\}\subseteq Y\subseteq E} \mu_X(e)\, m_X^{E\backslash\{e\}}(Y\backslash\{e\}) = \mu_X(e) \sum_{\varnothing\subseteq Y\subseteq E\backslash\{e\}} m_X^{E\backslash\{e\}}(Y)\\ &= \mu_X(e)\end{aligned}$$

■

Axiom Z-3 It holds that

$$\mathcal{F}^A\left(Q\widetilde{\square}\right) = \mathcal{F}^A(Q)\,\widetilde{\square} \quad n>0$$

For the proof of this axiom we will need the proofs of the properties of internal and external negation.

Proof of the property of external negation

We have to prove that

$$\mathcal{F}^A(\widetilde{\neg}Q) = \widetilde{\neg}\mathcal{F}^A(Q)$$

In the lemma 52 we have established that the induced negation operation is the standard negation.

Proof.

$$\begin{aligned}\mathcal{F}^A(\widetilde{\neg}Q)(X_1,\ldots,X_n) &= \sum_{Y_1\in\mathcal{P}(E)}\cdots\sum_{Y_n\in\mathcal{P}(E)} m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)\,(\widetilde{\neg}Q)(Y_1,\ldots,Y_n)\\ &= \sum_{Y_1\in\mathcal{P}(E)}\cdots\sum_{Y_n\in\mathcal{P}(E)} m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)\,(1-Q(Y_1,\ldots,Y_n))\\ &= \sum_{Y_1\in\mathcal{P}(E)}\cdots\sum_{Y_n\in\mathcal{P}(E)} m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)\\ &- \sum_{Y_1\in\mathcal{P}(E)}\cdots\sum_{Y_n\in\mathcal{P}(E)} Q(Y_1,\ldots,Y_n)\, m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)\\ &= 1-\mathcal{F}^A(Q)(X_1,\ldots,X_n) = \widetilde{\neg}\left(\mathcal{F}^A(Q)(X_1,\ldots,X_n)\right)\end{aligned}$$

■

Proof of the property of internal negation

We have to prove that

$$\mathcal{F}^A(Q\widetilde{\neg}) = \mathcal{F}^A(Q)\,\widetilde{\neg}$$

Lemma 56 *Let $X\in\widetilde{\mathcal{P}}(E)$ be a fuzzy set. Then*

$$m_X(Y) = m_{\widetilde{\neg}X}(\neg Y)$$

Proof.

$$m_X(Y) = \Pr(representative_X = Y) = \prod_{e\in Y}\mu_X(e)\prod_{e\in E\backslash Y}(1-\mu_X(e))$$
$$= \prod_{e\in Y}\widetilde{\neg}\widetilde{\neg}\mu_X(e)\prod_{e\in E\backslash Y}\widetilde{\neg}\mu_X(e) = \prod_{e\in E\backslash Y}\widetilde{\neg}\mu_X(e)\prod_{e\in Y}\widetilde{\neg}\widetilde{\neg}\mu_X(e)$$
$$= \prod_{e\in E\backslash Y}\mu_{\widetilde{\neg}X}(e)\prod_{e\in E\backslash(E\backslash Y)}\widetilde{\neg}\mu_{\widetilde{\neg}X}(e) = \prod_{e\in E\backslash Y}\mu_{\widetilde{\neg}X}(e)\prod_{e\in E\backslash(E\backslash Y)}(1-\mu_{\widetilde{\neg}X}(e))$$
$$= \Pr(representative_{\widetilde{\neg}X} = E\backslash Y) = m_{\widetilde{\neg}X}(\neg Y)$$

■

Proof of the property of internal negation:
Proof.

$$\mathcal{F}^A(Q\neg)(X_1,\ldots,X_n) = \sum_{Y_1\in\mathcal{P}(E)}\ldots\sum_{Y_n\in\mathcal{P}(E)} m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)(Q\neg)(Y_1,\ldots,Y_n)$$
$$= \sum_{Y_1\in\mathcal{P}(E)}\ldots\sum_{Y_n\in\mathcal{P}(E)} m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)\,Q(Y_1,\ldots,\neg Y_n)$$

and using the lemma 56 the above expression is equal to

$$\ldots = \sum_{Y_1\in\mathcal{P}(E)}\ldots\sum_{Y_n\in\mathcal{P}(E)} m_{X_1}(Y_1)\ldots m_{\widetilde{\neg}X_n}(\neg Y_n)\,Q(Y_1,\ldots,\neg Y_n)$$
$$\ldots = \sum_{Y_1\in\mathcal{P}(E)}\ldots\sum_{Y_n\in\mathcal{P}(E)} m_{X_1}(Y_1)\ldots m_{\widetilde{\neg}X_n}(Y_n)\,Q(Y_1,\ldots,Y_n)$$
$$\ldots = \mathcal{F}^A(Q)(X_1,\ldots,\widetilde{\neg}X_n)$$

■

Using the properties of internal and external negation duality is trivial.

$$\mathcal{F}^A(Q\square) = \mathcal{F}^A(\widetilde{\neg}Q\neg) = \widetilde{\neg}\mathcal{F}^A(Q)\,\widetilde{\neg}$$
$$= \mathcal{F}^A(Q)\,\widetilde{\square}$$

Axiom Z-4 It holds that

$$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\,\widetilde{\cup} \quad n>0 \tag{8}$$

In the proof of 8 we will use the following results:

Lemma 57 *Let $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ be given, $R \in \mathcal{P}(E)$ a crisp set. Then*

$$\sum_{Y_1\in\mathcal{P}(E)}\sum_{Y_2\in\mathcal{P}(E)/Y_1\cap Y_2=R} m_{X_1}(Y_1)\,m_{X_2}(Y_2) = m_{X_1\widetilde{\cap}X_2}(R) \tag{9}$$

where $X_1\widetilde{\cap}X_2$ is defined by means of the induced tnorm.

Proof. Let $E' = E \backslash R$. We will use 54.

It fulfills the following:

$$m_{X_1 \tilde{\cap} X_2}(R) = \prod_{e \in R} \mu_{X_1}(e)\,\mu_{X_2}(e) \prod_{e \in E \backslash R} \left(1 - \mu_{X_1}(e)\,\mu_{X_2}(e)\right) \tag{10}$$

As the sum 9 is restricted to the $Y_1, Y_2 \in \mathcal{P}(E)$ such that

$$Y_1 \cap Y_2 = R$$

then,

$$\begin{aligned} &\sum_{Y_1 \in \mathcal{P}(E)} \sum_{Y_2 \in \mathcal{P}(E) / Y_1 \cap Y_2 = R} m_{X_1}(Y_1)\, m_{X_2}(Y_2) \\ &= \sum_{R \subseteq Y_1 \subseteq E} \sum_{R \subseteq Y_2 \subseteq E / Y_1 \cap Y_2 = R} m_{X_1}(Y_1)\, m_{X_2}(Y_2) \\ &= \sum_{\varnothing \subseteq C_1 \subseteq E'} \sum_{\varnothing \subseteq C_2 \subseteq E' / C_1 \cap C_2 = \varnothing} m_{X_1}(C_1 \cup R)\, m_{X_2}(C_2 \cup R) \end{aligned} \tag{11}$$

and using lemma 54

$$\begin{aligned} &= \sum_{\varnothing \subseteq C_1 \subseteq E'} \sum_{\varnothing \subseteq C_2 \subseteq E' / C_1 \cap C_2 = \varnothing} m_{X_1}^{E'}(C_1) \prod_{e \in R} \mu_{X_1}(e)\, m_{X_2}^{E'}(C_2) \prod_{e \in R} \mu_{X_2}(e) \\ &= \prod_{e \in R} \mu_{X_1}(e)\,\mu_{X_2}(e) \sum_{\varnothing \subseteq C_1 \subseteq E'} \sum_{\varnothing \subseteq C_2 \subseteq E' / C_1 \cap C_2 = \varnothing} m_{X_1}^{E'}(C_1)\, m_{X_2}^{E'}(C_2) \\ &= \prod_{e \in R} \mu_{X_1}(e)\,\mu_{X_2}(e) \sum_{\varnothing \subseteq C_1 \subseteq E'} \sum_{C_1 \subseteq C_2 \subseteq E'} m_{X_1}^{E'}(C_1)\, m_{X_2}^{E'}(C_2 \backslash C_1) \end{aligned}$$

And the equality 10 will be fulfilled if:

$$\sum_{\varnothing \subseteq C_1 \subseteq E'} \sum_{C_1 \subseteq C_2 \subseteq E'} m_{X_1}^{E'}(C_1)\, m_{X_2}^{E'}(C_2 \backslash C_1) = \prod_{e \in E'} \left(1 - \mu_{X_1}(e)\,\mu_{X_2}(e)\right) \tag{12}$$

The proof is by induction in the cardinality of E'. We will denote by $E^i = \{e_1, \ldots, e_i\}$ a referential set with i elements.

Case $i = 1$ ($|E'| = 1$ ($E' = E^1 = \{e_1\}$):

$$\begin{aligned} &\sum_{\varnothing \subseteq C_1 \subseteq E'} \sum_{C_1 \subseteq C_2 \subseteq E'} m_{X_1}(C_1)\, m_{X_2}(C_2 \backslash C_1) \\ &= \sum_{\varnothing \subseteq C_1 \subseteq \{e_1\}} \sum_{C_1 \subseteq C_2 \subseteq \{e_1\}} m_{X_1}(C_1)\, m_{X_2}(C_2 \backslash C_1) \\ &= m_{X_1}(\varnothing)\left(m_{X_2}(\varnothing) + m_{X_2}(\{e_1\})\right) + m_{X_1}(\{e_1\})\, m_{X_2}(\varnothing) \\ &= \left(1 - \mu_{X_1}(e_1)\right)\left(1 - \mu_{X_2}(e_1) + \mu_{X_2}(e_1)\right) + \mu_{X_1}(e_1)\left(1 - \mu_{X_2}(e_1)\right) \\ &= 1 - \mu_{X_1}(e_1) + \mu_{X_1}(e_1)\left(1 - \mu_{X_2}(e_1)\right) \\ &= 1 - \mu_{X_1}(e_1) + \mu_{X_1}(e_1) - \mu_{X_1}(e_1)\,\mu_{X_2}(e_1) \\ &= 1 - \mu_{X_1}(e_1)\,\mu_{X_2}(e_1) \end{aligned}$$

Induction hypothesis: **Case** $i = m$ ($|E'| = m$, ($E' = E^m = \{e_1, \ldots, e_m\}$). We suppose that

$$\sum_{\varnothing \subseteq C_1 \subseteq E'} \sum_{C_1 \subseteq C_2 \subseteq E'} m_{X_1}(C_1)\, m_{X_2}(C_2 \backslash C_1) = \prod_{i=1}^{m} (1 - \mu_{X_1}(e_i)\, \mu_{X_2}(e_i))$$

Case $i = m + 1$: $|E'| = m + 1$, ($E' = E^{m+1} = \{e_1, \ldots, e_{m+1}\}$)
It should be noted that if $e_{m+1} \in C$ then

$$\begin{aligned} m_{X_1}(C) &= \prod_{e \in C} \mu_{X_1}(e) \prod_{e \notin C} (1 - \mu_{X_1}(e)) \\ &= \mu_{X_1}(e_{m+1}) \prod_{e \in C \backslash \{e_{m+1}\}} \mu_{X_1}(e) \prod_{e \notin C} (1 - \mu_{X_1}(e)) \\ &= \mu_{X_1}(e_{m+1})\, m_{X_1}^{E^m}(C \backslash \{e_{m+1}\}) \end{aligned}$$

Whilst if $e_{m+1} \notin C$ then we have

$$\begin{aligned} m_{X_1}(C) &= \prod_{e \in C} \mu_{X_1}(e) \prod_{e \notin C} (1 - \mu_{X_1}(e)) \\ &= (1 - \mu_{X_1}(e_{m+1})) \prod_{e \in C} \mu_{X_1}(e) \prod_{e \notin C \cup \{e_{m+1}\}} (1 - \mu_{X_1}(e)) \\ &= (1 - \mu_{X_1}(e_{m+1}))\, m_{X_1}^{E^m}(C) \end{aligned}$$

By computation

$$\sum_{\varnothing \subseteq C_1 \subseteq E^{m+1}} \sum_{C_1 \subseteq C_2 \subseteq E^{m+1}} m_{X_1}(C_1)\, m_{X_2}(C_2 \backslash C_1) \tag{13}$$

$$= \sum_{\varnothing \subseteq M_1 \subseteq E^m} \sum_{M_1 \subseteq M_2 \subseteq E^m} m_{X_1}(M_1)\, m_{X_2}(M_2 \backslash M_1) +$$

$$\sum_{\varnothing \subseteq M_1 \subseteq E^m} \sum_{M_1 \subseteq M_2 \subseteq E^m} m_{X_1}(M_1)\, m_{X_2}(M_2 \cup \{e_{m+1}\} \backslash M_1) +$$

$$\sum_{\varnothing \subseteq M_1 \subseteq E^m} \sum_{M_1 \subseteq M_2 \subseteq E^m} m_{X_1}(M_1 \cup \{e_{m+1}\})\, m_{X_2}(M_2 \cup \{e_{m+1}\} \backslash M_1 \cup \{e_{m+1}\})$$

As $C_1 \subseteq C_2 \subseteq E^{m+1}$ the situation in which $e_{m+1} \in C_1$ and $e_{m+1} \notin C_2$ is not possible.

Let we evaluate the three sums in 13. In the computation we use the induction hypothesis.

First sum:

$$\sum_{\varnothing\subseteq M_1\subseteq E^m}\sum_{M_1\subseteq M_2\subseteq E^m} m_{X_1}(M_1)\,m_{X_2}(M_2\backslash M_1) \tag{14}$$

$$= \sum_{\varnothing\subseteq M_1\subseteq E^m}\sum_{M_1\subseteq M_2\subseteq E^m} (1-\mu_{X_1}(e_{m+1}))\,m_{X_1}^{E^m}(M_1)\,(1-\mu_{X_2}(e_{m+1}))\,m_{X_2}^{E^m}(M_2\backslash M_1)$$

$$= (1-\mu_{X_1}(e_{m+1}))\,(1-\mu_{X_2}(e_{m+1}))\sum_{\varnothing\subseteq C_1\subseteq E^m}\sum_{C_1\subseteq C_2\subseteq E^m} m_{X_1}^{E^m}(C_1)\,m_{X_2}^{E^m}(C_2\backslash C_1)$$

$$= (1-\mu_{X_1}(e_{m+1}))\,(1-\mu_{X_2}(e_{m+1}))\prod_{i=1}^{m}(1-\mu_{X_1}(e_i)\,\mu_{X_2}(e_i))$$

Second sum:

$$\sum_{\varnothing\subseteq M_1\subseteq E^m}\sum_{M_1\subseteq M_2\subseteq E^m} m_{X_1}(M_1)\,m_{X_2}(M_2\cup\{e_{m+1}\}\backslash M_1) \tag{15}$$

$$= \sum_{\varnothing\subseteq M_1\subseteq E^m}\sum_{M_1\subseteq M_2\subseteq E^m} (1-\mu_{X_1}(e_{m+1}))\,m_{X_1}^{E^m}(M_1)\,\mu_{X_2}(e_{m+1})\,m_{X_2}^{E^m}(M_2\backslash M_1)$$

$$= (1-\mu_{X_1}(e_{m+1}))\,\mu_{X_2}(e_{m+1})\sum_{\varnothing\subseteq C_1\subseteq E^m}\sum_{C_1\subseteq C_2\subseteq E^m} m_{X_1}^{E^m}(C_1)\,m_{X_2}^{E^m}(C_2\backslash C_1)$$

$$= (1-\mu_{X_1}(e_{m+1}))\,\mu_{X_2}(e_{m+1})\prod_{i=1}^{m}(1-\mu_{X_1}(e_i)\,\mu_{X_2}(e_i))$$

Third sum:

$$\sum_{\varnothing\subseteq M_1\subseteq E^m}\sum_{M_1\subseteq M_2\subseteq E^m} m_{X_1}(M_1\cup\{e_{m+1}\})\,m_{X_2}(M_2\cup\{e_{m+1}\}\backslash M_1\cup\{e_{m+1}\}) \tag{16}$$

$$= \sum_{\varnothing\subseteq M_1\subseteq E^m}\sum_{M_1\subseteq M_2\subseteq E^m} \mu_{X_1}(e_{m+1})\,m_{X_1}^{E^m}(M_1)\,(1-\mu_{X_2}(e_{m+1}))\,m_{X_2}^{E^m}(M_2\backslash M_1)$$

$$= \mu_{X_1}(e_{m+1})\,(1-\mu_{X_2}(e_{m+1}))\prod_{i=1}^{m}(1-\mu_{X_1}(e_i)\,\mu_{X_2}(e_i))$$

And using expressions 14, 15 and 16:

$$\sum_{\varnothing\subseteq C_1\subseteq E^{m+1}} \sum_{C_1\subseteq C_2\subseteq E^{m+1}} m_{X_1}(C_1)\, m_{X_2}(C_2\backslash C_1)$$
$$= \ldots$$
$$= ((1-\mu_{X_1}(e_{m+1}))(1-\mu_{X_2}(e_{m+1})) + (1-\mu_{X_1}(e_{m+1})\mu_{X_2}(e_{m+1}) +$$
$$\mu_{X_1}(e_{m+1})(1-\mu_{X_2}(e_{m+1}))) \times \prod_{i=1}^{m}(1-\mu_{X_1}(e_i)\mu_{X_2}(e_i))$$
$$= ((1-\mu_{X_1}(e_{m+1})(1-\mu_{X_2}(e_{m+1}) + \mu_{X_2}(e_{m+1}))$$
$$\mu_{X_1}(e_{m+1})(1-\mu_{X_2}(e_{m+1}))) \times \prod_{i=1}^{m}(1-\mu_{X_1}(e_i)\mu_{X_2}(e_i))$$
$$= ((1-\mu_{X_1}(e_{m+1}) + \mu_{X_1}(e_{m+1}) - \mu_{X_1}(e_{m+1})\mu_{X_2}(e_{m+1}))$$
$$\times \prod_{i=1}^{m}(1-\mu_{X_1}(e_i)\mu_{X_2}(e_i))$$
$$= (1-\mu_{X_1}(e_{m+1})\mu_{X_2}(e_{m+1}))\prod_{i=1}^{m}(1-\mu_{X_1}(e_i)\mu_{X_2}(e_i))$$
$$= \prod_{i=1}^{m+1}(1-\mu_{X_1}(e_i)\mu_{X_2}(e_i))$$

In this way we have proved 12, and the lemma is satisfied. ■

Lemma 58 *Let* $Q : \mathcal{P}(E) \to \mathbf{I}$ *be a unary semi-fuzzy quantifier. Then it is fulfilled:*

$$\sum_{Y_1\in\mathcal{P}(E)} \sum_{Y_2\in\mathcal{P}(E)} m_{X_1}(Y_1)\, m_{X_2}(Y_2)\, Q(Y_1\cap Y_2) = \sum_{Y\in\mathcal{P}(E)} m_{X_1\tilde{\cap}X_2}(Y)\, Q(Y)$$

Proof. Using lemma 57:

$$\sum_{Y_1\in\mathcal{P}(E)} \sum_{Y_2\in\mathcal{P}(E)} m_{X_1}(Y_1)\, m_{X_2}(Y_2)\, Q(Y_1\cap Y_2)$$
$$= \sum_{R\in\mathcal{P}(E)} \sum_{Y_1\in\mathcal{P}(E)} \sum_{Y_2\in\mathcal{P}(E)/Y_1\cap Y_2=R} m_{X_1}(Y_1)\, m_{X_2}(Y_2)\, Q(R)$$
$$= \sum_{R\in\mathcal{P}(E)} Q(R) \sum_{Y_1\in\mathcal{P}(E)} \sum_{Y_2\in\mathcal{P}(E)/Y_1\cap Y_2=R} m_{X_1}(Y_1)\, m_{X_2}(Y_2)$$
$$= \sum_{R\in\mathcal{P}(E)} m_{X_1\tilde{\cap}X_2}(R)\, Q(R)$$
$$= \sum_{Y\in\mathcal{P}(E)} m_{X_1\tilde{\cap}X_2}(Y)\, Q(Y)$$

■

And finally we prove the fulfillment of expression 8:

Proof.

$$\mathcal{F}^A(Q\cap)(X_1,\dots,X_n,X_{n+1})$$
$$=\sum_{Y_1\in\mathcal{P}(E)}\dots\sum_{Y_{n+1}\in\mathcal{P}(E)} m_{X_1}(Y_1)\dots m_{X_{n+1}}(Y_{n+1})(Q\cap)(Y_1,\dots,Y_{n+1})$$
$$=\sum_{Y_1\in\mathcal{P}(E)}\dots\sum_{Y_{n+1}\in\mathcal{P}(E)} m_{X_1}(Y_1)\dots m_{X_{n+1}}(Y_{n+1})\,Q(Y_1,\dots,Y_n\cap Y_{n+1})$$
$$=\sum_{Y_1\in\mathcal{P}(E)}\dots\sum_{Y_{n-1}\in\mathcal{P}(E)} m_{X_1}(Y_1)\dots m_{X_{n-1}}(Y_{n-1})$$
$$\sum_{Y_n\in\mathcal{P}(E)}\sum_{Y_{n+1}\in\mathcal{P}(E)} m_{X_n}(Y_n)\,m_{X_{n+1}}(Y_{n+1})\,Q(Y_1,\dots,Y_{n-1},Y_n\cap Y_{n+1})$$
$$=\sum_{Y_1\in\mathcal{P}(E)}\dots\sum_{Y_{n-1}\in\mathcal{P}(E)} m_{X_1}(Y_1)\dots m_{X_{n-1}}(Y_{n-1})$$
$$\sum_{R\in\mathcal{P}(E)}\sum_{Y_n\in\mathcal{P}(E)}\sum_{Y_{n+1}\in\mathcal{P}(E)/Y_n\cap Y_{n+1}=R} m_{X_n}(Y_n)\,m_{X_{n+1}}(Y_{n+1})\,Q(Y_1,\dots,Y_{n-1,},R)$$
$$=\sum_{Y_1\in\mathcal{P}(E)}\dots\sum_{Y_{n-1}\in\mathcal{P}(E)} m_{X_1}(Y_1)\dots m_{X_{n-1}}(Y_{n-1})$$
$$\sum_{R\in\mathcal{P}(E)} m_{X_n\tilde{\cap}X_{n+1}}(R)\,Q(Y_1,\dots,Y_{n-1},R)$$

Where we have using 58. And then,

$$\mathcal{F}^A(Q\cap)(X_1,\dots,X_n,X_{n+1})$$
$$=\dots$$
$$=\sum_{Y_1\in\mathcal{P}(E)}\dots\sum_{Y_{n-1}\in\mathcal{P}(E)} m_{X_1}(Y_1)\dots m_{X_{n-1}}(Y_{n-1})$$
$$\sum_{R\in\mathcal{P}(E)} m_{X_n\tilde{\cap}X_{n+1}}(R)\,Q(Y_1,\dots,Y_{n-1},R)$$
$$=\mathcal{F}^A(Q)\left(X_1,\dots,X_n\widetilde{\cap}X_{n+1}\right)$$

■

Now we prove the fulfillment of the Z-4 axiom:

$$\mathcal{F}(Q\cup)=\mathcal{F}(Q)\,\widetilde{\cup},\quad n>0$$

Proof. For a semi-fuzzy quantifier $Q:\mathcal{P}(E)^n\to\mathbf{I}$ it is fulfilled (τ_n represents the trasposition of the n and $n+1$ element [34, section 4.5]):

$$\begin{aligned}
(Q\neg \cap \neg\tau_n\neg) &= (f' : (Y_1', \ldots, Y_n') \to Q\,(Y_1', \ldots, \neg Y_n')) \cap \neg\tau_n\neg \\
&= \left(f'' : \left(Y_1'', \ldots, Y_n'', Y_{n+1}''\right) \to f'\left(Y_1'', \ldots, Y_n'' \cap Y_{n+1}''\right)\right) \neg\tau_n\neg \\
&= \left(f'' : \left(Y_1'', \ldots, Y_n'', Y_{n+1}''\right) \to Q\left(Y_1'', \ldots, \neg\left(Y_n'' \cap Y_{n+1}''\right)\right)\right) \neg\tau_n\neg \\
&= \left(f''' : \left(Y_1''', \ldots, Y_n''', Y_{n+1}'''\right) \to f'' : \left(Y_1''', \ldots, Y_n''', \neg Y_{n+1}'''\right)\right) \tau_n\neg \\
&= \left(f''' : \left(Y_1''', \ldots, Y_n''', Y_{n+1}'''\right) \to Q : \left(Y_1''', \ldots, \neg\left(Y_n'' \cap \neg Y_{n+1}''\right)\right)\right) \tau_n\neg \\
&= \left(f'''' : \left(Y_1'''', \ldots, Y_n'''', Y_{n+1}''''\right) \to f''' : \left(Y_1'''', \ldots, Y_{n+1}'''', Y_n''''\right)\right) \neg \\
&= \left(f'''' : \left(Y_1'''', \ldots, Y_n'''', Y_{n+1}''''\right) \to Q : \left(Y_1'''', \ldots, \neg\left(Y_{n+1}'''' \cap \neg Y_n''''\right)\right)\right) \neg \\
&= \left(f''''' : \left(Y_1''''', \ldots, Y_n''''', Y_{n+1}'''''\right) \to f'''' : \left(Y_1'''', \ldots, Y_n'''', \neg Y_{n+1}''''\right)\right) \\
&= \left(f''''' : \left(Y_1''''', \ldots, Y_n''''', Y_{n+1}'''''\right) \to Q\left(Y_1''''', \ldots, \neg\left(\neg Y_n''''' \cap \neg Y_{n+1}'''''\right)\right)\right)
\end{aligned}$$

and then

$$\begin{aligned}
(Q\neg \cap \neg\tau_n\neg)\,(Y_1, \ldots, Y_n, Y_{n+1}) &= Q\,(Y_1, \ldots, Y_{n-1}, \neg\,(\neg Y_n \cap \neg Y_{n+1})) \\
&= Q\,(Y_1, \ldots, Y_{n-1}, Y_n \cup Y_{n+1})
\end{aligned}$$

In this way, we can use the properties of external negation, internal negation and trasposition of arguments (trivially fulfilled) and the expression 8 to obtain:

$$\begin{aligned}
\mathcal{F}^A\,(Q\cup)\,(X_1, \ldots, X_n, X_{n+1}) &= \mathcal{F}^A\,(Q\neg \cap \neg\tau_n\neg)\,(X_1, \ldots, X_n, X_{n+1}) \\
&= \mathcal{F}^A\,(Q)\,\widetilde{\neg}\widetilde{\cap}\widetilde{\neg}\tau_n\widetilde{\neg}\,(X_1, \ldots, X_n, X_{n+1}) \\
&= \mathcal{F}^A\,(Q)\left(X_1, \ldots, \widetilde{\neg}\left(\widetilde{\neg}X_n \widetilde{\cap} \widetilde{\neg}X_{n+1}\right)\right) \\
&= \mathcal{F}^A\,(Q)\left(X_1, \ldots, X_n \widetilde{\cup} X_{n+1}\right)
\end{aligned}$$

where in the last step we use that $\widetilde{\cap}$ and $\widetilde{\cup}$ are constructed by means of dual operators. ■

Axiom Z-5 It holds that

> If Q is nonincreasing in the n-th arg, then $\mathcal{F}(Q)$ is nonincreasing in the n-th arg, $n > 0$.

Proof. We will consider first the unary case.

Let $Q : \mathcal{P}(E) \to \mathbf{I}$ be a nonincreasing semi-fuzzy quantifier. We will proof that $\mathcal{F}^A(Q)(X)$ is nonincreasing using induction on the cardinality of the referential.

Case $i = 1$; that is, the referential contains only one element ($E = E^1 = \{e_1\}$).

Let $X, X' \in \widetilde{\mathcal{P}}(E)$ be fuzzy sets fulfilling $X \subseteq X'$. Note

$$\mu_X(e_1) \le \mu_{X'}(e_1)\,, 1 - \mu_X(e_1) \ge 1 - \mu_{X'}(e_1)$$

As $Q : \mathcal{P}(E) \to \mathbf{I}$ is monotonic nonincreasing for $h \geq 0$ it holds that

$$aQ(\varnothing) + bQ(\{e_1\}) \geq (a-h)\,Q(\varnothing) + (b+h)\,Q(\{e_1\}) \tag{17}$$

as a consequence of

$$(a-h)\,Q(\varnothing) + (b+h)\,Q(\{e_1\}) = aQ(\varnothing) + bQ(\{e_1\}) + h\,(Q(\{e_1\}) - Q(\varnothing))$$

and $Q(\{e_1\}) - Q(\varnothing) \leq 0$ because Q is nonincreasing.

Let be $h = \mu_{X'}(e_1) - \mu_X(e_1) \geq 0$. By 17

$$\begin{aligned}
&\mathcal{F}^A(Q)(X)\\
&= (1-\mu_X(e_1))\,Q(\varnothing) + \mu_X(e_1)\,Q(\{e_1\})\\
&\geq (1-\mu_X(e_1) - (\mu_{X'}(e_1) - \mu_X(e_1)))\,Q(\varnothing) + (\mu_X(e_1) + (\mu_{X'}(e_1) - \mu_X(e_1)))\,Q(\{e_1\})\\
&= (1-\mu_X(e_1) - \mu_{X'}(e_1) + \mu_X(e_1))\,Q(\varnothing) + (\mu_X(e_1) + \mu_{X'}(e_1) - \mu_X(e_1))\,Q(\{e_1\})\\
&= (1-\mu_{X'}(e_1))\,Q(\varnothing) + \mu_{X'}(e_1)\,Q(\{e_1\}) = \mathcal{F}^A(Q)(X')
\end{aligned}$$

and the property is fulfilled for a one referential set

Hypothesis of induction: Case $i = m$ ($E = E^m = \{e_1, \ldots, e_m\}$). For $X, X' \in \widetilde{\mathcal{P}}(E^m)$ such that $X \subseteq X'$ it holds that $\mathcal{F}^A(Q)(X) \geq \mathcal{F}^A(Q)(X')$.

Case $i = m+1$ ($E = E^{m+1} = \{e_1, \ldots, e_{m+1}\}$).

Based on $Q : \mathcal{P}(E^{m+1}) \to \mathbf{I}$ we define the semi-fuzzy quantifiers $Q^a : \mathcal{P}(E^m) \to \mathbf{I}$ and $Q^b : \mathcal{P}(E^m) \to \mathbf{I}$ as

$$\begin{aligned}
Q^a(X) &= Q(X),\, X \in \mathcal{P}(E^m)\\
Q^b(X) &= Q(X \cup \{e_{m+1}\}),\, X \in \mathcal{P}(E^m)
\end{aligned}$$

Q^a and Q^b are monotonic nonincreasing on E^m.

Let $h = (\mu_{X'}(e_{m+1}) - \mu_X(e_{m+1}))$. Then

$$\begin{aligned}
&\mathcal{F}^A(Q)(X)\\
&= \sum_{Y \in \mathcal{P}(E^{m+1})} m_X(Y)\,Q(Y)\\
&= \sum_{Y \in \mathcal{P}(E^m)} (1-\mu_X(e_{m+1}))\,m_X(Y)\,Q(Y) + \sum_{Y \in \mathcal{P}(E^m)} \mu_X(e_{m+1})\,m_X(Y)\,Q(Y \cup \{e_{m+1}\})\\
&\geq (1-\mu_X(e_{m+1}) - h) \sum_{Y \in \mathcal{P}(E^m)} m_X(Y)\,Q(Y) + (\mu_X(e_{m+1}) + h) \sum_{Y \in \mathcal{P}(E^m)} m_X(Y)\,Q(Y \cup \{e_{m+1}\})\\
&= (1-\mu_X(e_{m+1}) - (\mu_{X'}(e_{m+1}) - \mu_X(e_{m+1}))) \sum_{Y \in \mathcal{P}(E^m)} m_X(Y)\,Q(Y)\\
&+ (\mu_X(e_{m+1}) + (\mu_{X'}(e_{m+1}) - \mu_X(e_{m+1}))) \sum_{Y \in \mathcal{P}(E^m)} m_X(Y)\,Q(Y \cup \{e_{m+1}\})\\
&= (1-\mu_{X'}(e_{m+1})) \sum_{Y \in \mathcal{P}(E^m)} m_X(Y)\,Q(Y) + \mu_{X'}(e_{m+1}) \sum_{Y \in \mathcal{P}(E^m)} m_X(Y)\,Q(Y \cup \{e_{m+1}\})
\end{aligned}$$

And using the induction hypothesis

$$\sum_{Y\in\mathcal{P}(E^m)} m_X(Y)\,Q^a(Y) \geq \sum_{Y\in\mathcal{P}(E^m)} m_{X'}(Y)\,Q^a(Y)$$
$$\sum_{Y\in\mathcal{P}(E^m)} m_X(Y)\,Q^b(Y) \geq \sum_{Y\in\mathcal{P}(E^m)} m_{X'}(Y)\,Q^b(Y)$$

because $Q^a : \mathcal{P}(E^m) \to \mathbf{I}$ and $Q^b : \mathcal{P}(E^m) \to \mathbf{I}$ are monotonic nonincreasing on a referential of m elements. We continue the computation:

$$\begin{aligned}
&\mathcal{F}^A(Q)(X)\\
&\geq (1-\mu_{X'}(e_{m+1})) \sum_{Y\in\mathcal{P}(E^m)} m_X(Y)\,Q(Y) + \mu_{X'}(e_{m+1}) \sum_{Y\in\mathcal{P}(E^m)} m_X(Y)\,Q(Y\cup\{e_{m+1}\})\\
&\geq (1-\mu_{X'}(e_{m+1})) \sum_{Y\in\mathcal{P}(E^m)} m_{X'}(Y)\,Q^a(Y) + \mu_{X'}(e_{m+1}) \sum_{Y\in\mathcal{P}(E^m)} m_{X'}(Y)\,Q^b(Y)\\
&= \sum_{Y\in\mathcal{P}(E^m)} m_{X'}(Y)\,Q(Y)\\
&= \mathcal{F}^A(Q)(X')
\end{aligned}$$

Let us consider now the general case.Let $Q : \mathcal{P}(E)^n \to \mathbf{I}$ be an n-ary semi-fuzzy quantifier non increasing in its n argument. Then,

$$\begin{aligned}
&\mathcal{F}^A(Q)(X_1,\dots,X_n)\\
&= \sum_{Y_1\in\mathcal{P}(E)} \dots \sum_{Y_n\in\mathcal{P}(E)} m_{X_1}(Y_1)\dots m_{X_n}(Y_n)\,Q(Y_1,\dots,Y_n)\\
&= \sum_{Y_1\in\mathcal{P}(E)} \dots \sum_{Y_{n-1}\in\mathcal{P}(E)} m_{X_1}(Y_1)\dots m_{X_{n-1}}(Y_{n-1}) \sum_{Y_n\in\mathcal{P}(E)} m_{X_n}(Y_n)\,Q(Y_1,\dots,Y_n)
\end{aligned}$$

In

$$\sum_{Y_n\in\mathcal{P}(E)} m_{X_n}(Y_n)\,Q(Y_1,\dots,Y_n)$$

the $Y_1,\dots,Y_{n-1}$ are constant. The unary semi-fuzzy quantifier $Q' : \mathcal{P}(E) \to \mathbf{I}$

$$Q'(Y) = Q(Y_1,\dots,Y_{n-1},Y)$$

is monotonic non increasing, and then $\mathcal{F}^A(Q')$ is also monotonic non increasing. As this fact is fulfilled for all

$$Y_1,\dots,Y_{n-1} \in \mathcal{P}(E)$$

then the proposition is fulfilled. ■

Axiom Z-6 It holds that

$$\mathcal{F}\left(Q \circ \underset{i=1}{\overset{n}{\times}} \widehat{f}_i\right) = \mathcal{F}(Q) \circ \underset{i=1}{\overset{n}{\times}} \widehat{\mathcal{F}}(f_i) \text{ where } f_1, \ldots, f_n : E' \to E, E' \neq \varnothing \quad (18)$$

To prove this property we need some previous results.

Existential quantifier

Proposition 59 ($\mathcal{F}^A(\exists)(X)$) *Let $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy set. Then*

$$\mathcal{F}^A(\exists)(X) = \sup\left\{\underset{i=1}{\overset{m}{\widetilde{\vee}}} \mu_X(a_i) : A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E), a_i \neq a_j \text{ if } i \neq j\right\}$$

Proof. Let $E = \{e_1, \ldots, e_m\}$ be given. Using the lemma 55

$$\begin{aligned}
\mathcal{F}^A(\exists)(X) &= \sum_{Y \in \mathcal{P}(E)} m_X(Y)\, Q(Y) \\
&= \sum_{Y \in \mathcal{P}(E) | Y \neq \varnothing} m_X(Y) \\
&= 1 - m_X(\varnothing) \\
&= 1 - \prod_{i=1}^{m} (1 - \mu_X(e_i)) \\
&= \underset{i=1}{\overset{m}{\widetilde{\vee}}} \mu_X(e_i)
\end{aligned}$$

■

Universal quantifier

Proposition 60 ($\mathcal{F}^A(\forall)(X)$) *Let $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy set. Then*

$$\mathcal{F}^A(\forall)(X) = \inf\left\{\underset{i=1}{\overset{m}{\widetilde{\wedge}}} \mu_X(a_i) : A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E), a_i \neq a_j \text{ si } i \neq j\right\}$$

Proof. Let $E = \{e_1, \ldots, e_m\}$ be given. Then

$$\begin{aligned}
\mathcal{F}^A(\forall)(X) &= \sum_{Y \in \mathcal{P}(E)} m_X(Y)\, \forall(Y) \\
&= m_X(E) \\
&= \prod_{i=1}^{m} \mu_X(e_i)
\end{aligned}$$

■

Induced extension principle.

To compute the induce extension principle of the $\mathcal{F}^A$ we will use the definition 28.

Notation 61 **($\widetilde{\vee} : \widetilde{\mathcal{P}}(E) \to \mathbf{I}$)** *Let $E = \{e_1, \ldots, e_m\}$ and $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy set. By $\widetilde{\vee} : \widetilde{\mathcal{P}}(E) \to \mathbf{I}$ we will denote the generalization of the induced tconorm to fuzzy sets; that is,*

$$\widetilde{\vee}(X) = \begin{cases} \widetilde{\vee}\left(\mu_X(e_1), \widetilde{\vee}\left(\ldots, \widetilde{\vee}\left(\mu_X(e_{m-1}), \mu_X(e_m)\right)\right)\right) & : \quad m > 0 \\ 0 & : \quad m = 0 \end{cases}$$

Proposition 62 *Let $f : E \to E'$ (where $E, E' \neq \varnothing$). Let $e'_i \in E'$ be given and let $\widehat{f^{-1}}(e'_i) = \left\{e_{i_1}, \ldots e_{i_{k_i}}\right\}$ be the inverse image of e'_i. Then the induced extension principle of $\mathcal{F}^A$ for f is*

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(e'_i) = \begin{cases} \widetilde{\vee}\left(\left\{\mu_X(e_{i_1})/e_{i_1}, \ldots, \mu_X\left(e_{i_{k_i}}\right)/e_{i_{k_i}}\right\}\right) & : \quad k_i \geq 1 \\ 0 & : \quad \textit{otherwise} \end{cases} \tag{19}$$

$$= \mathcal{F}^A\left(\exists_{\widehat{f^{-1}}(e'_i)}\right)\left(X^{\widehat{f^{-1}}(e'_i)}\right)$$

where $\exists_{\widehat{f^{-1}}(e'_i)}$ represents the existential quantifier on the base universe $\widehat{f^{-1}}(e'_i)$ and $X^{\widehat{f^{-1}}(e'_i)}$ is the proyection of the fuzzy set X over $\widehat{f^{-1}}(e'_i)$.

Proof.

$$\begin{aligned} \mu_{\widehat{\mathcal{F}^A}(f)(X)}(e') &= \mathcal{F}^A\left(\chi_{\widehat{f}(\cdot)}(e')\right)(X) \\ &= \sum_{Y \in \mathcal{P}(E)} m_X(Y)\, \chi_{\widehat{f}(\cdot)}(e')(Y) \\ &= \sum_{Y \in \mathcal{P}(E)} m_X(Y)\, \chi_{\widehat{f}(Y)}(e') \\ &= \sum_{Y \in \mathcal{P}(E)} m_X(Y) \begin{cases} 0 & : \quad e' \notin \widehat{f}(Y) \\ 1 & : \quad e' \in \widehat{f}(Y) \end{cases} \\ &= \sum_{Y \in \mathcal{P}(E) \mid \exists e \in Y, f(e) = e'} m_X(Y) \end{aligned} \tag{20}$$

If $\widehat{f \cdot^{-1}}(e') = \varnothing$ the previous sum is 0 and we are in the second situation of expression 19.

Let us suppose $\widehat{f \cdot^{-1}}(e') \neq \varnothing$. ■

It should be noted that for each $Y \in \mathcal{P}(E)$ fulfilling $\exists e \in Y$, $f(e) = e'$ then the intersection of Y with the inverse image of e' ($\widehat{f \cdot^{-1}}(e')$) is not empty. As for all Y fulfilling this condition can be decomposed in the part intersecting with $\widehat{f \cdot^{-1}}(e')$, and the part that does not intersect with $\widehat{f \cdot^{-1}}(e')$ ($E \backslash \widehat{f}^{-1}(e')$)

expression 20 is equal to

$$
\begin{aligned}
\mu_{\widehat{\mathcal{F}^A}(f)(X)}(e') &= \dots \\
&= \sum_{\varnothing\subset M\subseteq \widehat{f}^{-1}(e')} \sum_{\varnothing\subseteq R\subseteq E\backslash \widehat{f}^{-1}(e')} m_X(M\cup R) \\
&= \sum_{\varnothing\subset M\subseteq \widehat{f}^{-1}(e')} \sum_{\varnothing\subseteq R\subseteq E\backslash \widehat{f}^{-1}(e')} m_X^{\widehat{f}^{-1}(e')}(M)\, m_X^{E\backslash \widehat{f}^{-1}(e')}(R) \\
&= \sum_{\varnothing\subset M\subseteq \widehat{f}^{-1}(e')} m_X^{\widehat{f}^{-1}(e')} \sum_{\varnothing\subseteq R\subseteq E\backslash \widehat{f}^{-1}(e')} (M)\, m_X^{E\backslash \widehat{f}^{-1}(e')}(R) \\
&= \sum_{\varnothing\subset M\subseteq \widehat{f}^{-1}(e')} m_{X_1}^{\widehat{f}^{-1}(e')}(M)\cdot 1
\end{aligned}
$$

As $\varnothing \subseteq R \subseteq E\backslash \widehat{f}^{-1}(e')$ contains all the sets of $E\backslash \widehat{f}^{-1}(e')$. And then,

$$
\begin{aligned}
\mu_{\widehat{\mathcal{F}^A}(f)(X)}(e') &= \dots \\
&= \sum_{\varnothing\subset M\subseteq \widehat{f}^{-1}(e')} m_X^{\widehat{f}^{-1}(e')}(M) \\
&= \mathcal{F}^A\left(\exists_{\widehat{f}^{-1}(e')}\right)\left(X^{\widehat{f}^{-1}(e')}\right)
\end{aligned}
$$

If we denote $\widehat{f^{-1}}(e') = \{e_{i_1}, \dots e_{i_k}\}$ using expression 59 we obtain

$$\mu_{\widehat{\mathcal{F}^A}(f)(X)}(e') = \widetilde{\vee}\left(\{\mu_X(e_{i_1})/e_{i_1}, \dots, \mu_X(e_{i_k})/e_{i_k}\}\right)$$

Now we will prove the fulfillment of

$$\mathcal{F}\left(Q \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i\right) = \mathcal{F}(Q) \circ \mathop{\times}_{i=1}^{n} \widehat{\mathcal{F}}(f_i)$$
where $f_1, \dots, f_n : E' \to E, E' \neq \varnothing$

Proof. Let $E = \{e_1, \dots, e_m\}$, $E' = \{e_1, \dots, e_{m'}\}$ finite sets, $Q : \mathcal{P}(E)^n \to \mathbf{I}$ a semi-fuzzy quantifier, $X'_1 \dots, X'_n \in \mathcal{P}(E')$ fuzzy sets and $f_1, \dots, f_n : E' \to E, E' \neq \varnothing$. We point that $\widehat{\mathcal{F}}(f_i)(X_i) \in \widetilde{\mathcal{P}}(E)$. By computation

$$
\begin{aligned}
&\mathcal{F}^A(Q) \circ \times_{i=1}^{n} \widehat{\mathcal{F}^A}(f_i)(X'_1, \dots, X'_n) \qquad (21)\\
&= \mathcal{F}^A(Q) \circ \left(\widehat{\mathcal{F}^A}(f_1)(X'_1), \dots, \widehat{\mathcal{F}^A}(f_n)(X'_n)\right) \\
&= \sum_{Y_1\in\mathcal{P}(E)} \dots \sum_{Y_n\in\mathcal{P}(E)} m_{\widehat{\mathcal{F}^A}(f_1)(X'_1)} \dots m_{\widehat{\mathcal{F}^A}(f_n)(X'_n)} Q(Y_1, \dots, Y_n)
\end{aligned}
$$

Using result 62 we can rewrite $\widehat{\mathcal{F}^A}(f_i)(X'_i) \in \widetilde{\mathcal{P}}(E)$ as

$$
\begin{aligned}
&\widehat{\mathcal{F}^A}(f_i)(X'_i) \\
&= \left\{\mathcal{F}^A\left(\exists_{\widehat{f_i}^{-1}(e_1)}\right)\left((X'_i)^{\widehat{f_i}^{-1}(e_1)}\right)/e_1, \dots, \mathcal{F}^A\left(\exists_{\widehat{f_i}^{-1}(e_m)}\right)\left((X'_i)^{\widehat{f_i}^{-1}(e_m)}\right)/e_m\right\}
\end{aligned}
$$

Rewriting expression 21

$$\ldots = \sum_{Y_1 \in \mathcal{P}(E)} \cdots \sum_{Y_n \in \mathcal{P}(E)} \tag{22}$$

$$m_{\left\{\mathcal{F}^A\left(\exists_{\widehat{f_1}^{-1}(e_1)}\right)\left((X_1')^{\widehat{f_1}^{-1}(e_1)}\right)/e_1,\ldots,\mathcal{F}^A\left(\exists_{\widehat{f_1}^{-1}(e_m)}\right)\left((X_1')^{\widehat{f_1}^{-1}(e_m)}\right)/e_m\right\}}(Y_1)\ldots$$

$$m_{\left\{\mathcal{F}^A\left(\exists_{\widehat{f_n}^{-1}(e_1)}\right)\left((X_n')^{\widehat{f_n}^{-1}(e_1)}\right)/e_1,\ldots,\mathcal{F}^A\left(\exists_{\widehat{f_n}^{-1}(e_m)}\right)\left((X_n')^{\widehat{f_n}^{-1}(e_m)}\right)/e_m\right\}}(Y_n)\,Q(Y_1,\ldots,Y_n)$$

Let be$Y_j = \left\{e_{j_1},\ldots,e_{j_k}\right\} \in \mathcal{P}(E), E\backslash Y_j = \left\{e_{j_{k+1}},\ldots,e_{j_m}\right\}$. We will compute the probability mass $m_{\widehat{\mathcal{F}^A}(f_i)(X_i')}(Y_j)$:

$$m_{\widehat{\mathcal{F}^A}(f_i)(X_i')}(Y_j)$$

$$= m_{\left\{\mathcal{F}^A\left(\exists_{\widehat{f_i}^{-1}(e_1)}\right)\left((X_i')^{\widehat{f_i}^{-1}(e1)}\right)/e_1,\ldots,\mathcal{F}^A\left(\exists_{\widehat{f_i}^{-1}(e_m)}\right)\left((X_i')^{\widehat{f_i}^{-1}(e_m)}\right)/e_m\right\}}\left(\left\{e_{j_1},\ldots,e_{j_k}\right\}\right)$$

$$= \mathcal{F}^A\left(\exists_{\widehat{f_i}^{-1}\left(e_{j_1}\right)}\right)\left((X_i')^{\widehat{f_i}^{-1}\left(e_{j_1}\right)}\right)\cdot\ldots\cdot\mathcal{F}^A\left(\exists_{\widehat{f_i}^{-1}\left(e_{j_k}\right)}\right)\left((X_i')^{\widehat{f_i}^{-1}\left(e_{j_k}\right)}\right)\cdot$$

$$\left(1-\mathcal{F}^A\left(\exists_{\widehat{f_i}^{-1}\left(e_{j_{k+1}}\right)}\right)\left((X_i')^{\widehat{f_i}^{-1}\left(e_{j_{k+1}}\right)}\right)\right)\cdot\ldots\cdot\left(1-\mathcal{F}^A\left(\exists_{\widehat{f_i}^{-1}\left(e_{j_m}\right)}\right)\left((X_i')^{\widehat{f_i}^{-1}\left(e_{j_m}\right)}\right)\right)$$

And by duality we now that $\widetilde{\neg}\mathcal{F}^A(\exists) = \mathcal{F}^A(\forall)\,\widetilde{\neg}$. Then,,

$$m_{\widehat{\mathcal{F}^A}(f_i)(X_i')}(Y_j) \tag{23}$$

$$= \ldots$$

$$= \mathcal{F}^A\left(\exists_{\widehat{f_i}^{-1}\left(e_{j_1}\right)}\right)\left((X_i')^{\widehat{f_i}^{-1}\left(e_{j_1}\right)}\right)\cdot\ldots\cdot\mathcal{F}^A\left(\exists_{\widehat{f_i}^{-1}\left(e_{j_k}\right)}\right)\left((X_i')^{\widehat{f_i}^{-1}\left(e_{j_k}\right)}\right)\cdot$$

$$\mathcal{F}\left(\forall_{\widehat{f_i}^{-1}\left(e_{j_{k+1}}\right)}\right)\left(\widetilde{\neg}\,(X_i')^{\widehat{f_i}^{-1}\left(e_{j_{k+1}}\right)}\right)\cdot\ldots\cdot\mathcal{F}^A\left(\forall_{\widehat{f_i}^{-1}\left(e_{j_m}\right)}\right)\left(\widetilde{\neg}\,(X_i')^{\widehat{f_i}^{-1}\left(e_{j_m}\right)}\right)$$

As

$$\begin{aligned}
\mathcal{F}^A\left(\exists_{\widehat{f_i}^{-1}\left(e_{j_r}\right)}\right)\left((X_i')^{\widehat{f_i}^{-1}\left(e_{j_r}\right)}\right) &= \sum_{\varnothing \subset M \subseteq \widehat{f}_i^{-1}\left(e_{j_r}\right)} m^{\widehat{f_i}^{-1}\left(e_{j_r}\right)}_{(X_i')^{\widehat{f_i}^{-1}\left(e_{j_r}\right)}}(M) \\
&= \sum_{\varnothing \subset M \subseteq \widehat{f}_i^{-1}\left(e_{j_r}\right)} m^{\widehat{f_i}^{-1}\left(e_{j_r}\right)}_{X_i'}(M) \\
\mathcal{F}^A\left(\forall_{\widehat{f_i}^{-1}\left(e_{j_r}\right)}\right)\left(\widetilde{\neg}\,(X_i')^{\widehat{f_i}^{-1}\left(e_{j_r}\right)}\right) &= \prod_{e' \in \widehat{f_i}^{-1}\left(e_{j_r}\right)} \widetilde{\neg}\mu_{(X_i')^{\widehat{f_i}^{-1}\left(e_{j_r}\right)}}(e') \\
&= \prod_{e' \in \widehat{f_i}^{-1}\left(e_{j_r}\right)} \widetilde{\neg}\mu_{X_i'}(e')
\end{aligned}$$

then expression 23 is equivalent to

$$
\begin{aligned}
& m_{\widehat{\mathcal{F}^A}(f_i)\left(X_i'\right)}\left(Y_j\right) \qquad (24)\\
& = \ldots \\
& = \sum_{\varnothing \subset M \subseteq \widehat{f}_i^{-1}\left(e_{j1}\right)} m_{X_i'}^{\widehat{f}_i^{-1}\left(e_{j1}\right)}(M) \cdot \ldots \cdot \sum_{\varnothing \subset M \subseteq \widehat{f}_i^{-1}\left(e_{j_k}\right)} m_{X_i'}^{\widehat{f}_i^{-1}\left(e_{j_k}\right)}(M) \\
& \cdot \prod_{e' \in E' \backslash \left(\widehat{f}_i^{-1}\left(e_{j1}\right) \cup \ldots \cup \widehat{f}_r^{-1}\left(e_{j_k}\right)\right)} \widetilde{\neg} \mu_{X_i}\left(e'\right) \\
& = \sum_{\substack{M \in \mathcal{P}\left(E'\right) \mid \\ \widehat{f}_i^{-1}\left(e_{j1}\right) \cap Y_j \neq \varnothing \wedge \ldots \wedge \\ \widehat{f}_i^{-1}\left(e_{j_k}\right) \cap Y_j \neq \varnothing \wedge \\ \widehat{f}_r^{-1}\left(e_{j_{k+1}}\right) \cap Y_j = \varnothing \wedge \ldots \wedge \\ \widehat{f}_r^{-1}\left(e_{j_m}\right) \cap Y_j = \varnothing}} m_{X_i'}(M)
\end{aligned}
$$

In this way, the probability mass $m_{\widehat{\mathcal{F}^A}(f_i)\left(X_i'\right)}\left(Y_j\right)$ is computed by using the probability masses $m_{X_i'}(M)$ that are associated to the $Ms \in \mathcal{P}(E')$ such that the intersection with the inverse image of the $e \in Y_j$ is not empty, and such that the intersection with the $e \in E \backslash Y_j$ is empty. Moreover, we should note that all $M \in \mathcal{P}(E')$ is associated to one $Y \in \mathcal{P}(E)$; that is, $\widehat{f}_i(M) = Y$ for some Y. In this way, if the Ys visit $\mathcal{P}(E)$, then the Ms visit $\mathcal{P}(E')$, and continuing with the computation of expression 22 we obtain

$$\mathcal{F}^A(Q)\circ\times_{i=1}^n\widehat{\mathcal{F}^A}(f_i)(X'_1,\dots,X'_n)$$
$$=\sum_{Y_1\in\mathcal{P}(E)}\dots\sum_{Y_n\in\mathcal{P}(E)} m_{\widehat{\mathcal{F}^A}(f_1)(X'_1)}\dots m_{\widehat{\mathcal{F}^A}(f_n)(X'_n)}(Y_1,\dots,Y_n)$$
$$=\dots$$
$$=\sum_{Y_1\in\mathcal{P}(E)}\dots\sum_{Y_n\in\mathcal{P}(E)} m_{\left\{\mathcal{F}^A\left(\exists_{\widehat{f_1}^{-1}(e_1)}\right)\left(\left(X'_1\right)^{\widehat{f_1}^{-1}(e_1)}\right)/e_1,\dots,\mathcal{F}^A\left(\exists_{\widehat{f_1}^{-1}(e_m)}\right)\left(\left(X'_1\right)^{\widehat{f_1}^{-1}(e_m)}\right)/e_m\right\}}(Y_1)$$
$$\dots m_{\left\{\mathcal{F}^A\left(\exists_{\widehat{f_n}^{-1}(e_1)}\right)\left((X'_n)^{\widehat{f_n}^{-1}(e_1)}\right)/e_1,\dots,\mathcal{F}^A\left(\exists_{\widehat{f_n}^{-1}(e_m)}\right)\left((X'_n)^{\widehat{f_n}^{-1}(e_m)}\right)/e_m\right\}}(Y_n)$$
$$Q(Y_1,\dots,Y_n)$$
$$=\dots$$
$$=\sum_{\substack{Y_1=\left\{e_{(Y_1)_1},\dots,e_{(Y_1)_k}\right\}\in\mathcal{P}(E),\\ E\backslash Y_1=\left\{e_{(Y_1)_{k+1}},\dots,e_{(Y_1)_m}\right\}}}\dots\sum_{\substack{Y_n=\left\{e_{(Y_n)_1},\dots,e_{(Y_n)_k}\right\}\in\mathcal{P}(E),\\ E\backslash Y_n=\left\{e_{(Y_n)_{k+1}},\dots,e_{(Y_n)_m}\right\}}}$$
$$\sum_{\substack{Y'_1\in\mathcal{P}(E')|\\ \widehat{f}_1^{-1}\left(e_{(Y_1)_1}\right)\cap Y'_1\neq\varnothing\wedge\dots\wedge\\ \widehat{f}_1^{-1}\left(e_{(Y_1)_k}\right)\cap Y'_1\neq\varnothing\wedge\\ \widehat{f}_1^{-1}\left(e_{(Y_1)_{k+1}}\right)\cap Y'_1=\varnothing\wedge\dots\wedge\\ \widehat{f}_1^{-1}\left(e_{(Y_1)_m}\right)\cap Y'_1=\varnothing}} m_{X'_1}(Y'_1)\dots\sum_{\substack{Y'_n\in\mathcal{P}(E')|\\ \widehat{f}_n^{-1}\left(e_{(Y_n)_1}\right)\cap Y'_n\neq\varnothing\wedge\dots\wedge\\ \widehat{f}_n^{-1}\left(e_{(Y_n)_k}\right)\cap Y'_n\neq\varnothing\wedge\\ \widehat{f}_n^{-1}\left(e_{(Y_n)_{k+1}}\right)\cap Y'_n=\varnothing\wedge\dots\wedge\\ \widehat{f}_n^{-1}\left(e_{(Y_n)_m}\right)\cap Y'_n=\varnothing}} m_{X'_n}(Y'_n)\,Q(Y_1,\dots,Y_n)$$

$$=\sum_{\substack{Y_1=\left\{e_{(Y_1)_1},\dots,e_{(Y_1)_k}\right\}\in\mathcal{P}(E),\\ E\backslash Y_1=\left\{e_{(Y_1)_{k+1}},\dots,e_{(Y_1)_m}\right\}}}\dots\sum_{\substack{Y_n=\left\{e_{(Y_n)_1},\dots,e_{(Y_n)_k}\right\}\in\mathcal{P}(E),\\ E\backslash Y_n=\left\{e_{(Y_n)_{k+1}},\dots,e_{(Y_n)_m}\right\}}}$$
$$\sum_{\substack{Y'_1\in\mathcal{P}(E')|\\ \widehat{f}_1^{-1}\left(e_{(Y_1)_1}\right)\cap Y'_1\neq\varnothing\wedge\dots\wedge\\ \widehat{f}_1^{-1}\left(e_{(Y_1)_k}\right)\cap Y'_1\neq\varnothing\wedge\\ \widehat{f}_1^{-1}\left(e_{(Y_1)_{k+1}}\right)\cap Y'_1=\varnothing\wedge\dots\wedge\\ \widehat{f}_1^{-1}\left(e_{(Y_1)_m}\right)\cap Y'_1=\varnothing}} m_{X'_1}(Y'_1)\dots\sum_{\substack{Y'_n\in\mathcal{P}(E')|\\ \widehat{f}_n^{-1}\left(e_{(Y_n)_1}\right)\cap Y'_n\neq\varnothing\wedge\dots\wedge\\ \widehat{f}_n^{-1}\left(e_{(Y_n)_k}\right)\cap Y'_n\neq\varnothing\wedge\\ \widehat{f}_n^{-1}\left(e_{(Y_n)_{k+1}}\right)\cap Y'_n=\varnothing\wedge\dots\wedge\\ \widehat{f}_n^{-1}\left(e_{(Y_n)_m}\right)\cap Y'_n=\varnothing}} m_{X'_n}(Y'_n)\,Q\left(\widehat{f}_1(Y'_1),\dots,\widehat{f}_n(Y'_n)\right)$$
$$=\sum_{Y'_1\in\mathcal{P}(E')} m_{X'_1}(Y'_1)\dots\sum_{Y'_n\in\mathcal{P}(E')} m_{X'_n}(Y'_n)\,Q\left(\widehat{f}_1(Y'_1),\dots,\widehat{f}_n(Y'_n)\right)$$
$$=\sum_{Y'_1\in\mathcal{P}(E')} m_{X'_1}(Y'_1)\dots\sum_{Y'_n\in\mathcal{P}(E')} m_{X'_n}(Y'_n)\left(Q\circ\times_{i=1}^n\widehat{f}_i\right)(Y'_1,\dots,Y'_n)$$
$$=\mathcal{F}\left(Q\circ\times_{i=1}^n\widehat{f}_i\right)(X'_1,\dots,X'_n)$$

as we want to prove. ■

8.2 Properties of the $\mathcal{F}^A$ that are not consequences of the DFS framework

8.2.1 Property of argument continuity

The $\mathcal{F}^A$ model fulfills the property of argument continuity.
Proof. In [17, appendix A] a detailed proof can be found. But the next arguments are enough to prove continuity. Let us consider the definition of the $\mathcal{F}^A$ model:

$$\mathcal{F}^A(Q)(X_1,\ldots,X_n) = \sum_{Y_1\in\mathcal{P}(E)} \cdots \sum_{Y_n\in\mathcal{P}(E)} m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)\, Q(Y_1,\ldots,Y_n)$$

Note that for crisp sets $Y_1,\ldots,Y_n \in \mathcal{P}(E)$ $m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)\, Q(Y_1,\ldots,Y_n)$ are continuous funtions, because $Q(Y_1,\ldots,Y_n)$ is constant and $m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)$ only involves the use of the product operation.

But the sum of continuous functions (that is, the sum over $(Y_1,\ldots,Y_n) \in \mathcal{P}(E)^n$ is continuous. And then the model is continuous in arguments. ■

8.2.2 Propery of quantifier continuity

The model $\mathcal{F}^A$ fulfills the property of Q-continuity:
Proof. Let $Q, Q' : \mathcal{P}(E)^n \to \mathbf{I}$ semi-fuzzy quantifiers. Then,

$$\begin{aligned}
& d\left(\mathcal{F}^A(Q), \mathcal{F}^A\left(Q'\right)\right) \\
&= \sup\left\{\left| \begin{array}{c} \sum_{Y_1\in\mathcal{P}(E)} \cdots \sum_{Y_n\in\mathcal{P}(E)} m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)\, Q(Y_1,\ldots,Y_n) \\ -\sum_{Y_1\in\mathcal{P}(E)} \cdots \sum_{Y_n\in\mathcal{P}(E)} m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)\, Q'(Y_1,\ldots,Y_n) \end{array} \right| \right. \\
& \left. : X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)\right\} \\
&= \sup\left\{\left| \begin{array}{c} \sum_{Y_1\in\mathcal{P}(E)} \cdots \sum_{Y_n\in\mathcal{P}(E)} m_{X_1}(Y_1)\ldots m_{X_n}(Y_n) \\ (Q(Y_1,\ldots,Y_n) - Q'(Y_1,\ldots,Y_n)) \end{array} \right| : X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)\right\} \\
&\le \sup\left\{\left| \sum_{Y_1\in\mathcal{P}(E)} \cdots \sum_{Y_n\in\mathcal{P}(E)} m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)\, d\left(Q, Q'\right)\right| : X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)\right\} \\
&= \sup\left\{\left| d\left(Q, Q'\right) \sum_{Y_1\in\mathcal{P}(E)} \cdots \sum_{Y_n\in\mathcal{P}(E)} m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)\right| : X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)\right\} \\
&= \sup\left\{\left| d\left(Q, Q'\right)\right| : X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)\right\} \\
&= d\left(Q, Q'\right)
\end{aligned}$$

And the property is fulfilled for $\delta < \varepsilon$. ■

8.2.3 Property of the fuzzy argument insertion

The $\mathcal{F}^A$ verifies the property of fuzzy argument insertion.

Proof. Let $Q : \mathcal{P}^{n+1}(E) \to \mathbf{I}$ a semi-fuzzy quantifier be given and $A \in \widetilde{\mathcal{P}}(E)$ a fuzzy set. For all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$ crisp we have

$$
\begin{aligned}
& Q\widetilde{\lhd}A(Y_1,\ldots,Y_n)\\
&= \mathcal{U}\left(\mathcal{F}^A(Q) \lhd A\right)(Y_1,\ldots,Y_n)\\
&= \mathcal{U}\left(\left(\begin{array}{c} f:(K_1,\ldots,K_{n+1}) \in \widetilde{\mathcal{P}}(E) \to \sum_{Z_1\in\mathcal{P}(E)}\cdots\sum_{Z_n\in\mathcal{P}(E)}\sum_{Z_{n+1}\in\mathcal{P}(E)} \\ m_{K_1}(Z_1)\ldots m_{K_n}(Z_n)\, m_{K_{n+1}}(Z_{n+1})\, Q(Z_1,\ldots,Z_n,Z_{n+1}) \end{array}\right) \lhd A\right)\\
&(Y_1,\ldots,Y_n)\\
&= \mathcal{U}\left(\left(f':(K'_1,\ldots,K'_n) \in \widetilde{\mathcal{P}}(E) \to f(K'_1,\ldots,K'_n,A)\right)\right)(Y_1,\ldots,Y_n)\\
&= \mathcal{U}\left(\left(\begin{array}{c} f':(K'_1,\ldots,K'_n) \in \widetilde{\mathcal{P}}(E) \to \sum_{Z_1\in\mathcal{P}(E)}\cdots\sum_{Z_n\in\mathcal{P}(E)}\sum_{Z_{n+1}\in\mathcal{P}(E)} \\ m_{K'_1}(Z_1)\ldots m_{K'_n}(Z_n)\, m_A(Z_{n+1})\, Q(Z_1,\ldots,Z_n,Z_{n+1}) \end{array}\right)\right)\\
&(Y_1,\ldots,Y_n)\\
&= \left(\begin{array}{c} f'':(K''_1,\ldots,K''_n) \in \mathcal{P}(E) \to \sum_{Z_1\in\mathcal{P}(E)}\cdots\sum_{Z_n\in\mathcal{P}(E)}\sum_{Z_{n+1}\in\mathcal{P}(E)} \\ m_{K''_1}(Z_1)\ldots m_{K''_n}(Z_n)\, m_A(Z_{n+1})\, Q(Z_1,\ldots,Z_n,Z_{n+1}) \end{array}\right)\\
&(Y_1,\ldots,Y_n)\\
&= \left(f'':(K''_1,\ldots,K''_n) \in \mathcal{P}(E) \to \sum_{Z_{n+1}\in\mathcal{P}(E)} m_A(Z_{n+1})\, Q(K''_1,\ldots,K''_n,Z_{n+1})\right)\\
&(Y_1,\ldots,Y_n)
\end{aligned}
$$

because $(K''_1,\ldots,K''_n) \in \mathcal{P}(E)$ are crisp sets, and then

$$
\begin{aligned}
Q\widetilde{\lhd}A(Y_1,\ldots,Y_n) &= \ldots \\
&= \sum_{Z_{n+1}\in\mathcal{P}(E)} m_A(Z_{n+1})\, Q(Y_1,\ldots,Y_n,Z_{n+1})
\end{aligned} \tag{25}
$$

Using the previously obtained result (expression 25), then:

$$
\begin{aligned}
& \mathcal{F}^A\left(Q\widetilde{\lhd}A\right)(X_1,\ldots,X_n) \qquad (26)\\
&= \sum_{Y_1\in\mathcal{P}(E)}\cdots\sum_{Y_n\in\mathcal{P}(E)} m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)\left(Q\widetilde{\lhd}A\right)(Y_1,\ldots,Y_n)\\
&= \sum_{Y_1\in\mathcal{P}(E)}\cdots\sum_{Y_n\in\mathcal{P}(E)} m_{X_1}(Y_1)\ldots m_{X_n}(Y_n) \sum_{Z_{n+1}\in\mathcal{P}(E)} m_A(Z_{n+1})\, Q(Y_1,\ldots,Y_n,Z_{n+1})\\
&= \sum_{Y_1\in\mathcal{P}(E)}\cdots\sum_{Y_n\in\mathcal{P}(E)}\sum_{Y_{n+1}\in\mathcal{P}(E)} m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)\, m_A(Y_{n+1})\, Q(Y_1,\ldots,Y_n,A)\\
&= \mathcal{F}^A(Q)(X_1,\ldots,X_n,A)\\
&= \mathcal{F}^A(Q) \lhd A
\end{aligned}
$$

■

8.2.4 Property of the identity quantifier.

This property is easily proved by induction. Let us denote

$$\Pr\left(card_X = j\right) = \sum_{Y \in \mathcal{P}(E) || Y| = j} m_X\left(Y\right)$$

the probability that the cardinality of a crisp representative of X let be j.
Proof. For $|E| = m$ we have

$$\begin{aligned}
\mathcal{F}^A\left(\mathbf{identity}\right)(X) &= \sum_{Y \in \mathcal{P}(E)} m_X\left(Y\right) \mathbf{identity}\left(Y\right) \\
&= \sum_{Y \in \mathcal{P}(E)} m_X\left(Y\right) \frac{|Y|}{|E|} \\
&= \sum_{j=0}^{m} \sum_{Y \in \mathcal{P}(E) || Y| = j} m_X\left(Y\right) \frac{j}{m} \\
&= \frac{1}{m} \sum_{j=0}^{m} j \Pr\left(card_X = j\right)
\end{aligned}$$

Let us begin the induction proof:
Case $i = 1$, $X \in \widetilde{\mathcal{P}}\left(E^1\right)$. Evident.
Induction hypothesis. Case $i = m$ (that is, $E = E^m = \{e_1, \ldots, e_m\}$). For $X \in \widetilde{\mathcal{P}}\left(E\right)$ it is fulfilled

$$\mathcal{F}^A\left(\mathbf{identity}\right)(X) = \frac{1}{m} \sum_{j=1}^{m} \mu_X\left(e_j\right), X \in \widetilde{\mathcal{P}}\left(E\right)$$

Casw $i = m + 1$ ($E = E^{m+1} = \{e_1, \ldots, e_{m+1}\}$).
For an m elements referential is fulfilled (using the induction hypothesis).

$$\sum_{j=0}^{m} \Pr\left(card_X = j\right) \frac{j+1}{m+1} \tag{27}$$

$$\begin{aligned}
&= \Pr\left(card_X = 0\right) \frac{1}{m+1} + \Pr\left(card_X = 1\right) \frac{2}{m+1} + \ldots + \Pr\left(card_X = m\right) \frac{m+1}{m+1} \\
&= \Pr\left(card_X = 0\right) \frac{0}{m+1} + \Pr\left(card_X = 1\right) \frac{1}{m+1} + \ldots + \Pr\left(card_X = m\right) \frac{m}{m+1} + \\
&\Pr\left(card_X = 0\right) \frac{1}{m+1} + \Pr\left(card_X = 1\right) \frac{1}{m+1} + \ldots + \Pr\left(card_X = m\right) \frac{1}{m+1} \\
&= \sum_{j=0}^{m} \Pr\left(card_X = j\right) \frac{j}{m+1} + \frac{1}{m+1} \sum_{j=0}^{m} \Pr\left(card_X = j\right)
\end{aligned}$$

$$= \sum_{j=0}^{m} \Pr\left(card_X = j\right) \frac{j}{m+1} + \frac{1}{m+1}$$

$$= \frac{m}{m+1} \sum_{j=0}^{m} \Pr\left(card_X = m\right) \frac{j}{m} + \frac{1}{m+1}$$

$$= \frac{m}{m+1} \mathcal{F}^A\left(\textbf{identity}\right)(X) + \frac{1}{m+1}$$

$$= \frac{m}{m+1} \frac{1}{m} \sum_{j=0}^{m} \mu_X\left(e_j\right) + \frac{1}{m+1}$$

$$= \frac{1}{m+1} \sum_{j=0}^{m} \mu_X\left(e_j\right) + \frac{1}{m+1}$$

Let us suppose now that $X \in \widetilde{\mathcal{P}}\left(E^{m+1}\right)$. And let be $X' \in \widetilde{\mathcal{P}}\left(E^{m}\right)$ the fuzzy set

$$X' = X^{E^m}$$

Then,

$\mathcal{F}^A\left(\textbf{identity}\right)(X)$

$$= \sum_{j=0}^{m+1} \Pr\left(card_X = j\right) \frac{j}{m+1}$$

$$= \Pr\left(card_{X'} = 0\right)\left(1 - \mu_X\left(e_{m+1}\right)\right) \frac{0}{m+1} +$$

$$+ \sum_{j=1}^{m} \left(\Pr\left(card_{X'} = j\right)\left(1 - \mu_X\left(e_{m+1}\right)\right) + \Pr\left(card_{X'} = j-1\right) \mu_X\left(e_{m+1}\right)\right) \frac{j}{m+1}$$

$$+ \Pr\left(card_{X'} = m\right) \mu_X\left(e_{m+1}\right) \frac{m+1}{m+1}$$

$$= \sum_{j=1}^{m} \left(\Pr\left(card_{X'} = j-1\right) \mu_X\left(e_{m+1}\right)\right) \frac{j}{m+1} + \Pr\left(card_{X'} = m\right) \mu_X\left(e_{m+1}\right) \frac{m+1}{m+1}$$

$$+ \sum_{j=1}^{m} \Pr\left(card_{X'} = j\right)\left(1 - \mu_X\left(e_{m+1}\right)\right) \frac{j}{m+1} + \Pr\left(card_{X'} = 0\right)\left(1 - \mu_X\left(e_{m+1}\right)\right) \frac{0}{m+1}$$

$$= \sum_{j=1}^{m+1} \left(\Pr\left(card_{X'} = j-1\right) \mu_X\left(e_{m+1}\right)\right) \frac{j}{m+1} + \sum_{j=0}^{m} \Pr\left(card_{X'} = j\right)\left(1 - \mu_X\left(e_{m+1}\right)\right) \frac{j}{m+1}$$

$$= \mu_X\left(e_{m+1}\right) \sum_{j=0}^{m} \Pr\left(card_{X'} = j\right) \frac{j+1}{m+1} + \left(1 - \mu_X\left(e_{m+1}\right)\right) \sum_{j=0}^{m} \Pr\left(card_{X'} = j\right) \frac{j}{m+1}$$

$$= \mu_X\left(e_{m+1}\right) \sum_{j=0}^{m} \Pr\left(card_{X'} = j\right) \frac{j+1}{m+1} + \left(1 - \mu_X\left(e_{m+1}\right)\right) \frac{m}{m+1} \sum_{j=0}^{m} \Pr\left(card_{X'} = j\right) \frac{j}{m}$$

And using expression 27 and the induction hypothesis:

$$= \mu_X(e_{m+1})\left(\frac{1}{m+1}\sum_{j=0}^{m}\mu_X(e_j)+\frac{1}{m+1}\right)+(1-\mu_X(e_{m+1}))\frac{m}{m+1}\frac{1}{m}\sum_{j=0}^{m}\mu_X(e_j)$$

$$= \mu_X(e_{m+1})\left(\frac{1}{m+1}\sum_{j=0}^{m}\mu_X(e_j)+\frac{1}{m+1}\right)+(1-\mu_X(e_{m+1}))\frac{1}{m+1}\sum_{j=0}^{m}\mu_X(e_j)$$

$$= \mu_X(e_{m+1})\frac{1}{m+1}\sum_{j=0}^{m}\mu_X(e_j)+\mu_X(e_{m+1})\frac{1}{m+1}$$

$$+\frac{1}{m+1}\sum_{j=0}^{m}\mu_X(e_j)-\mu_X(e_{m+1})\frac{1}{m+1}\sum_{j=0}^{m}\mu_X(e_j)$$

$$= \mu_X(e_{m+1})\frac{1}{m+1}+\frac{1}{m+1}\sum_{j=0}^{m}\mu_X(e_j)$$

$$= \frac{1}{m+1}\left(\sum_{j=0}^{m}\mu_X(e_j)+\mu_X(e_{m+1})\right)$$

$$= \frac{1}{m+1}\sum_{j=0}^{m+1}\mu_X(e_j)$$

■

8.2.5 Property of the probabilistic interpretation of quantifiers

The model $\mathcal{F}^A$ fulfills the property of the probabilistic interpretation of quantifiers:

Proof. Let $Q_1,\ldots,Q_r:\mathcal{P}^n(E)\to\mathbf{I}$ a probabilistic covering of the quantification universe. Then for all $X_1,\ldots,X_n\in\widetilde{\mathcal{P}}(E)$

$$\mathcal{F}(Q_1)(X_1,\ldots,X_n)+\ldots+\mathcal{F}(Q_r)(X_1,\ldots,X_n)$$

$$=\sum_{Y_1\in\mathcal{P}(E)}\ldots\sum_{Y_n\in\mathcal{P}(E)}m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)\,Q_1(Y_1,\ldots,Y_n)+\ldots+$$

$$+\sum_{Y_1\in\mathcal{P}(E)}\ldots\sum_{Y_n\in\mathcal{P}(E)}m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)\,Q_r(Y_1,\ldots,Y_n)$$

$$=\sum_{Y_1\in\mathcal{P}(E)}\ldots\sum_{Y_n\in\mathcal{P}(E)}m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)\left(Q_1(Y_1,\ldots,Y_n)+\ldots+Q_r(Y_1,\ldots,Y_n)\right)$$

$$=\sum_{Y_1\in\mathcal{P}(E)}\ldots\sum_{Y_n\in\mathcal{P}(E)}m_{X_1}(Y_1)\ldots m_{X_n}(Y_n)$$

$$=1$$

■

9 Apendix B. Efficient computation of the $\mathcal{F}^A$ model

Although the time to compute the result of evaluatign a quantified expression could seem extremely high, it is possible to develop polynomial algorithms for quantitative quantifiers[19]. In table 1 the algorithm to evaluate unary quantitative quantifiers is shown. A quantitative unary semi-fuzzy quantifier depends on a function $q : \{0, \ldots, |E|\} \to \mathbf{I}$; that is, a function of the possible cardinality values in $\mathbf{I}$. The idea of the algorithm is that, if we know the probabilities of the cardinalities in a base set of size k; that is, we know the probabilities $\Pr(card_X = 0), \ldots, \Pr(card_X = k)$ then when we add one element to the base set (e_{k+1}) we have to consider two possibilities to compute the change in the probabilities of the cardinalities. One possibility is that the element e_{k+1} fulfills the property represented by X (with probability $\mu_X(e_{k+1})$) and the other is that the element does not fulfill the property represented by X (with probability $(1 - \mu_X(e_{k+1}))$). The next formula expresses the change in the probabilities:

$$\Pr(card_X = j) = \begin{cases} \Pr(card_X = 0)(1 - \mu_X(e_{k+1})) & : \quad j = 0 \\ \Pr(card_X = j)(1 - \mu_X(e_{k+1})) \\ + \Pr(card_X = j-1)\,\mu_X(e_{k+1}) & : \quad 1 \le j \le k \\ \Pr(card_X = m)\,\mu_X(e_{i+1}) & : \quad j = k+1 \end{cases}$$

Similar ideas can be used to develop algorithms for other quantitative quantifiers. The case of binary proportional quantifiers can be consulted in [17, pag. 348].

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[19] Quantitative quantifiers are invariant under automorphims [34, section 4.13]. Quantitative quantifiers can be expressed as a function of the cardinalities of their arguments and their boolean combinations.

Algorithm for computing $\mathcal{F}^A(Q)(X)$

```
INPUT: X[0,...,m-1], m ≥ 1,q:{0,...,m}→I
double pr_aux_i,pr_aux_i_minus_1;
double pr[0,....,m];
result = 0;
pr[0] = 1;
for (j = 0;j < m;j++) {
    pr_aux_i = pr[0];
    pr[0] = (1 - X[j]) × pr_aux_i;
    pr_aux_i_minus_1 = pr_aux_i;
    for (i = 1; i <= j;i++) {
        pr_aux_i = pr[i];
        pr[i] = (1 - X[j]) × pr_aux_i + X[j] × pr_aux_i_minus_1;
        pr_aux_i_minus_1 = pr_aux_i;
    }
    pr[j+1] = X[j] × pr_aux_i_minus_1;
}
for (j = 0;j <= m;j++)
    result = result + pr[j] × q(j);
return result;
```

Table 1: Algorithm for computing unary quantitative quantifiers $\mathcal{F}^A(Q)(X)$

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